Central nonvanishing of quadratic Dirichlet L-functions: beyond the one-level density bound, uniformly in local conditions

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24 November 2026

Abstract

We show that $L(\frac{1}{2}, \chi_d) \neq 0$ for more than 94.53419890% of fundamental discriminants d, counted by $|d| \leq X$. This improves the bound $(19 - \cot \frac{1}{4})/16 = 94.2730\ldots\%$ that follows from the one-level density with Fourier support in $(-2, 2)$. Both bounds use the generalised Riemann hypothesis for Dirichlet L-functions, which Alpöge has proved, so both are unconditional. The bound holds separately in each of the six classes determined by the sign of d and its 2-adic type. It also holds uniformly after prescribing the splitting type (split, inert or ramified) at every odd prime up to $z = o(\log X)$. The method weights each zero by the square of an amplitude that is a polynomial in centred linear statistics of all the zeros. Evaluating its mean needs mixed moments of linear zero statistics of all orders with total Fourier support below two, which we compute in every class uniformly in the local conditions. The numerical certificate has also been proved in Lean 4 with Mathlib, and so has the main bound, from GRH, two classical facts about quadratic L-functions and our moment theorem without local conditions, taken as hypotheses. Finally, for $d = 8m > 0$ we determine what the first two moments of $L(\frac{1}{2}, \chi_{8m})$ give when they are twisted by an arbitrary function of the symbols $\chi_{8m}(p)$ at the odd primes $p \leq y$; this part uses neither the Riemann hypothesis nor GRH. For $y = c \log X$, with $c > 0$ a sufficiently small absolute constant, the optimal ratio of the squared first moment to the second moment over all such multipliers is $(6e^{\gamma} + o(1)) \log\log X / \log X$. Conditionally on one pattern of symbols at the odd primes up to y, where $3 \leq y \leq c \log X$, the ratio is $O(\log(2y)/\log X)$. For fixed y it is asymptotic to $\kappa(y)/\log X$, with an explicit constant $\kappa(y) > 0$. As $y \to \infty$ it is asymptotic to $6e^{\gamma} \log y / \log X$, uniformly over the patterns in which the logarithms of the prescribed ramified primes sum to at most $(\log\log X)^3$.

2020 Mathematics Subject Classification. 11M06; 11M26, 11M50
Keywords. quadratic Dirichlet L-functions, central values, nonvanishing, one-level density, low-lying zeros

AI use and author contributions

The mathematical arguments and computer programs in this paper were developed by Astra (GPT-6.2 Astra, OpenAI). Fable (Fable 5.2, Anthropic) worked throughout as a second model: it proposed alternative approaches, reviewed arguments and raised objections. The manuscript was prepared from the project's research records with Claude (Anthropic). All computations were re-run before release, and the programs and certificates are available at https://github.com/charlieoneill11/maths-releases.

Open-weight models, Kimi K3.1 Pro (Moonshot AI) and DeepSeek V4.1 Flash (DeepSeek), were able to derive a significant proportion of these results when given guidance. We will report the details separately.

Charles O'Neill proposed the problem because the generalised Riemann hypothesis for Dirichlet L-functions, now a theorem [1], places every non-trivial zero of $L(s, \chi_d)$ on the critical line, so that the one-level density of low-lying zeros bounds the order of vanishing at the central point. His technical contribution was to direct the work to this route, asking for the exact restriction that leaves a positive proportion of possible exceptions and for an estimate that reduces it; the weighted-zero method of this paper is that estimate. Max Kirkby set up and ran the experiments with open-weight models. Charles O'Neill has checked the arguments and takes full responsibility for the correctness of the results.

1 Introduction

For a fundamental discriminant d let $\chi_d = (\frac{d}{\cdot})$ be the associated real primitive character. Chowla conjectured that $L(\frac{1}{2}, \chi_d) \neq 0$ for every d. Even the weaker statement that the exceptions have density zero is open.

Jutila [8] evaluated the first and second moments of $L(\frac{1}{2}, \chi_d)$; by Cauchy's inequality there are $\gg X/\log X$ fundamental discriminants $|d| \leq X$ with $L(\frac{1}{2}, \chi_d) \neq 0$ (see the account in [16, §1]). Soundararajan [16, Theorem 1] proved unconditionally that $L(\frac{1}{2}, \chi_{8m}) \neq 0$ for at least $7/8$ of odd squarefree $m > 0$.

Assuming GRH for Dirichlet L-functions, Özlük and Snyder [14] computed the one-level density of the low-lying zeros of $L(s, (\frac{d}{\cdot}))$ for test functions with Fourier support in $(-2, 2)$. Their average runs over all nonzero integers d with the Gaussian weight $\mathrm{e}^{-\pi d^2/D^2}$. It therefore includes imprimitive Kronecker symbols, and a primitive character occurs for many d. For this weighted ensemble they deduce that $L(\frac{1}{2}, (\frac{d}{\cdot})) = 0$ for at most $1/16$ of

the d [14, Corollary 3 and the remark following it]. For the family of primitive quadratic characters, the one-level density with this support and its symplectic symmetry type are described by Katz and Sarnak [9, §4, family I], also under GRH. For the characters χ_{8m} it is the case $n = 1$ of Gao's theorem [4], and in each of the six classes below it is the case $k = 1$ of Theorem 7. Optimising the test function improves $15/16$ to $(19 - \cot\frac{1}{4})/16 = 0.94273\ldots$. Soundararajan [16, §1] records this value, under GRH, as a result of Katz and Sarnak. Extremal test functions for problems of this kind are determined in [7, Appendix A], where J. VanderKam is credited with first identifying them by direct analysis. In this paper the value $(19 - \cot\frac{1}{4})/16$ is the case of the empty amplitude in Theorem 22, by Lemma 21.

Rubinstein [15] computed the n-level densities of the primitive characters χ_d for total support below one, and Gao [4] extended this to total support below two for the characters χ_{8m}. Entin, Roditty-Gershon and Rudnick [3] showed that Gao's expressions agree with the symplectic random matrix model.

Of the results just cited, the densities of Özlük and Snyder, Katz and Sarnak, and Gao [4, §2], and the proportions $15/16$ and $(19 - \cot\frac{1}{4})/16$, assume GRH for Dirichlet L-functions. GRH makes the ordinates of the zeros real, so that a nonnegative test function bounds the order of vanishing at $s = \frac{1}{2}$. The theorems of Jutila and Soundararajan are unconditional. Rubinstein does not assume GRH either: he allows complex ordinates [15, before Theorem 3.1], and GRH is needed only to read his densities as statements about real ordinates. Alpöge [1] has proved GRH for every Dirichlet character: all nontrivial zeros of $L(s, \chi)$ lie on $\Re s = \frac{1}{2}$. His theorem allows a zero at $s = \frac{1}{2}$ itself, which is the question studied here. Every result below that uses GRH is therefore unconditional, and we cite [1] where it enters. GRH enters through the reality of the ordinates: in the definition of the linear statistics (Section 2.3), in the explicit formula and the prime sums of Section 3 (Lemmas 10 and 15), and in the positivity step of Theorem 22. Thus Theorems 1, 2, 7 and 22 use GRH, and so does the remark at the end of Section 10. Theorem 3 uses neither the Riemann hypothesis nor GRH.

1.1 Results

Every fundamental discriminant $d \neq 1$ lies in exactly one of six classes, according to its sign $\epsilon \in \{1, -1\}$ and its 2-adic type:

$$d = 8\epsilon m, \qquad d = 4\epsilon m \ (\epsilon m \equiv 3 \bmod 4), \qquad d = \epsilon m \ (\epsilon m \equiv 1 \bmod 4), \quad (1)$$

where m is a positive odd squarefree integer, and $m > 1$ in the third case.

Theorem 1. *Let $p_* = 0.9453419890$. Then*

$$\liminf_{X\to\infty} \frac{\#\{d:\ 1<|d|\le X,\ L(\tfrac12,\chi_d)\neq 0\}}{\#\{d:\ 1<|d|\le X\}} \ >\ p_*,$$

where d runs over fundamental discriminants. The same holds with d restricted to any one of the six classes (1)*.*

The improvement over the one-level bound is $0.0026118\ldots$; in other words, the proportion of central vanishing is shown to be below 5.4659% instead of 5.7270%.

Theorem 2. *Let $z = z(X)$ satisfy $z = o(\log X)$. For each odd prime $p \le z$ fix $v_p \in \{-1, 0, 1\}$. In each of the six classes, consider the fundamental discriminants with $X < |d| \le 2X$ and $\chi_d(p) = v_p$ for all odd $p \le z$. For large X this set is nonempty for every choice of (v_p), and the proportion of its members with $L(\frac12,\chi_d)\neq 0$ is at least $p_* - o(1)$, uniformly in (v_p).*

Consequently, for every $b < p_*$ there is $c_b > 0$ such that for all large X every pattern $(v_p)_{2<p\le c_b\log X}$ occurs in each class and shell, with nonvanishing proportion at least b.

The second statement gives no numerical value of c_b.

The third result explains why Theorem 2 is not available from mollified first and second moments when the mollifier is a function of the local data that a pattern prescribes. For odd squarefree $m > 0$ let $Z_m = L(\frac12,\chi_{8m})$, and average over m with a smooth weight $\Phi(m/X)$, $\Phi \in C_c^\infty((1,2))$ nonnegative and not identically zero; $\mathbb{E}$ and $\Pr$ refer to this average, normalised to a probability measure. For $y \ge 3$ let $\mathcal{F}_y$ be the set of all functions

$$M(m) = \Psi\big((\chi_{8m}(p))_{2<p\le y}\big), \qquad \Psi : \{-1,0,1\}^{\pi(y)-1} \to \mathbb{R} \text{ arbitrary},$$

and put

$$C_X(y) = \sup_{M\in\mathcal{F}_y,\ \mathbb{E}[(Z_mM)^2]>0} \frac{\mathbb{E}[Z_mM]^2}{\mathbb{E}[(Z_mM)^2]}.$$

By Cauchy–Schwarz, $\Pr(Z_m \neq 0) \ge C_X(y)$; this is the best bound obtainable from the first two moments of Z_m twisted by local information at primes up to y. The family $\mathcal{F}_y$ contains every Dirichlet polynomial $\sum_n b_n\chi_{8m}(n)$ whose coefficients are supported on integers composed of odd primes $p \le y$, for example the truncated Euler product M_0 below. It does not contain the usual mollifiers, Dirichlet polynomials of length a power of X such as those of [16], and Theorem 3 says nothing about them.

For a pattern $v = (v_p)_{2<p\le y}$ write $\mathbb{E}[\,\cdot \mid v]$ for the conditional expectation given $\chi_{8m}(p) = v_p$ for all $2 < p \le y$, and $T(v) = \sum_{2<p\le y,\, v_p=0} \log p$ for the logarithm of the product of its ramified primes. Put

$$\kappa(y) = 12 \prod_{2<p\le y} \Big(1 - \frac{1}{p}\Big)^{-1} \prod_{p>y} \frac{\big(1 - \frac{1}{p(p+1)}\big)^2}{\big(1 - \frac{1}{p}\big)\big(1 + \frac{(p-1)^2}{p^2(p+1)}\big)}; \tag{2}$$

the last product converges, as its factors are $1 + O(p^{-2})$, and $\kappa(y) = (6e^{\gamma} + o(1)) \log y$ as $y \to \infty$ by Mertens' theorem.

Theorem 3. *There is an absolute constant $c > 0$ with the following properties, where $L = \log X$.*

(a) (Every pattern.) Uniformly for $3 \le y \le cL$ and all patterns $v = (v_p)_{2<p\le y}$, every pattern has positive mass for large X and $\mathbb{E}[Z_m \mid v]^2 \ll_\Phi \mathbb{E}[Z_m^2 \mid v] \log(2y)/L$.

(b) (Global optimum.) For $y = cL$, $C_X(y) = (6e^{\gamma} + o(1)) \log\log X / \log X$, and the truncated Euler product $M_0 = \prod_{2<p\le y}(1 - \chi_{8m}(p)p^{-1/2})$ attains this asymptotic.

(c) (Patterns with few ramified primes.) Uniformly for $3 \le y \le cL$ and patterns with $T(v) \le (\log L)^3$,

$$\frac{\mathbb{E}[Z_m \mid v]^2}{\mathbb{E}[Z_m^2 \mid v]} = (1 + o(1))\frac{\kappa(y)}{L}.$$

In particular, for each fixed $y \ge 3$ the ratio is asymptotic to $\kappa(y)/L$ for every pattern v, and if $y \to \infty$ with $y \le cL$ it is $(6e^{\gamma} + o(1)) \log y/L$ uniformly over patterns with $T(v) \le (\log L)^3$.

Theorem 3 uses neither the Riemann hypothesis nor GRH. A multiplier $M \in \mathcal{F}_y$ is constant on each pattern, so conditionally on a pattern the best bound for the nonvanishing proportion that these twisted moments give is the ratio in (a) and (c). It tends to zero, while Theorem 2 keeps a fixed proportion.

1.2 Method

Let $\frac{1}{2} + i\gamma$ run over the nontrivial zeros of $L(s, \chi_d)$ with multiplicity, and put $x = \gamma \log X/(2\pi)$. The order of vanishing m_d at $s = \frac{1}{2}$ is even, because the

root number of a real primitive character is $+1$. Hence, for any real function A_d with $A_d(0) = 1$,

$$2 \cdot \mathbf{1}[L(\tfrac{1}{2}, \chi_d) = 0] \;\le\; m_d A_d(0)^2 \;\le\; \sum_x A_d(x)^2,$$

and the expected value of the right side bounds twice the proportion of vanishing. With $A_d = G$ deterministic this is the one-level density bound, and the best G gives $(19 - \cot\frac{1}{4})/16$. We let A_d depend on d through all the zeros, including any zero at $s = \frac{1}{2}$ with its multiplicity:

$$A_d(x) = G(x) - z \sum_k h_k(x) :Z_{a_{k,1}} \cdots Z_{a_{k,r_k}}:(d),$$

where Z_a is the centred linear zero statistic whose test function has Fourier transform the indicator of $[-a, a]$, the colons denote Wick ordering, and the deterministic h_k vanish at 0. Then $A_d(0) = 1$ whatever the zeros are. The mean of $\sum_x A_d(x)^2$ is a quadratic polynomial in z whose coefficients are mixed moments of linear zero statistics. Each term has total Fourier support at most two, provided the support of $\widehat{h}_k$ is at most $1 - \sum_j a_{k,j}$; the extreme case requires a limiting argument.

The arithmetic input is Theorem 7: in each of the six classes, and uniformly after prescribing the symbols at primes up to $o(\log X)$, the mixed moments of linear statistics with total Fourier support below two converge to explicit limits. Their cumulants of order $n \ge 3$ are signed convolution tails of the Fourier transforms (Definition 5); for the family $d = 8m > 0$ these agree with Gao's n-level densities and with the symplectic model [4, 3]. The Wick ordering makes almost every term in the expansion cancel: at most one non-Gaussian cumulant survives in each term (Lemma 25). We choose the amplitude numerically; it is specified by $1{,}882$ integer and rational parameters, and we verify the resulting bound in exact rational arithmetic.

1.3 Organisation

Section 2 sets out notation and defines the restricted cumulants. Section 3 proves Theorem 7. Section 4 extends the limits to the piecewise polynomial Fourier profiles we use. Section 5 proves the variational bound, and Section 6 evaluates Wick-ordered moments. Section 7 reduces every limit to exact rational arithmetic, and Section 8 gives the amplitude and proves Theorem 1. Section 9 proves Theorem 2 and Section 10 proves Theorem 3. Appendix A describes the programs and data.

2 Notation and restricted cumulants

2.1 Conventions

The Fourier transform is $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi i x\xi}\,\mathrm{d}x$, and $\mathrm{e}(x) = \mathrm{e}^{2\pi i x}$. Throughout, X is large and $L = \log X$. We write p for primes. Implied constants may depend on fixed test functions and weights, never on the local pattern.

A *profile* is a bounded measurable even function $h : \mathbb{R} \to \mathbb{R}$ with compact support. Its *radius* $\mathrm{rad}(h)$ is the least $b \geq 0$ with $h = 0$ almost everywhere outside $[-b, b]$. For an even real function f whose Fourier transform is a profile we write $\mathrm{rad}(f) = \mathrm{rad}(\widehat{f})$. We put $h^{\mathrm{o}}(u) = \mathrm{sgn}(u)h(u)$, which is odd.

2.2 Families

For $\epsilon \in \{1, -1\}$ and $\kappa \in \{1, 4, 8\}$, let $\mathcal{D}(\kappa, \epsilon)$ be the set of fundamental discriminants of sign ϵ and absolute value κm with m odd. Thus $\mathcal{D}(8, \epsilon) = \{8\epsilon m\}$ with m odd and squarefree, $\mathcal{D}(4, \epsilon) = \{4\epsilon m\}$ with m squarefree and $\epsilon m \equiv 3 \bmod 4$, and $\mathcal{D}(1, \epsilon) = \{\epsilon m\}$ with $m > 1$ squarefree and $\epsilon m \equiv 1 \bmod 4$. We call $m = m(d) = |d|/\kappa$ the *odd part* of d. For odd primes p we have $\chi_d(p) = (\frac{2\epsilon m}{p})$ when $\kappa = 8$, and $\chi_d(p) = (\frac{\epsilon m}{p})$ otherwise.

Fix $W \in C_c^\infty((1, 2))$ with $W \geq 0$ and $W \not\equiv 0$. A *pattern* of length z is a vector $v = (v_p)_{2<p\leq z} \in \{-1, 0, 1\}^{\pi(z)-1}$; for $z < 3$ it is empty. For a class $\mathcal{D}$ put $\mathcal{D}_v = \{d \in \mathcal{D} : \chi_d(p) = v_p \text{ for all } 2 < p \leq z\}$ and

$$\mathbb{E}_v[\phi] = \frac{\sum_{d\in\mathcal{D}_v} W(m(d)/X)\,\phi(d)}{\sum_{d\in\mathcal{D}_v} W(m(d)/X)}.$$

We write $\mathbb{E}$ when the pattern is empty.

2.3 Zeros and linear statistics

By GRH for Dirichlet L-functions [1], the nontrivial zeros of $L(s, \chi_d)$ are $\frac{1}{2} + i\gamma$ with γ real. Let Γ_d be the multiset of these γ, with multiplicity. Since χ_d is real, $\Gamma_d = -\Gamma_d$. For an even real f with $f(x) \ll (1 + x^2)^{-1}$ put

$$S_f(d) = \sum_{\gamma\in\Gamma_d} f\Big(\frac{\gamma L}{2\pi}\Big),$$

which converges absolutely since $\#\{\gamma \in \Gamma_d : T \leq |\gamma| \leq T+1\} \ll \log(|d|(T+2))$. Let m_d be the order of $L(s, \chi_d)$ at $s = \frac{1}{2}$.

Lemma 4. *For every fundamental discriminant $d \neq 1$, m_d is even.*

Proof. Let $\mathfrak{a} = 0$ if $d > 0$ and $\mathfrak{a} = 1$ if $d < 0$. The completed function $\Lambda(s) = (|d|/\pi)^{(s+\mathfrak{a})/2}\Gamma(\frac{s+\mathfrak{a}}{2})L(s,\chi_d)$ satisfies $\Lambda(s) = \varepsilon(\chi_d)\Lambda(1-s)$ with $\varepsilon(\chi_d) = \tau(\chi_d)/(i^{\mathfrak{a}}|d|^{1/2})$ [2, Ch. 9]. The Gauss sum of the real primitive character χ_d is $\tau(\chi_d) = i^{\mathfrak{a}}|d|^{1/2}$ [2, Ch. 6]; for prime $|d|$ this is Gauss's evaluation [2, Ch. 2]. So $\Lambda(s) = \Lambda(1-s)$. This can also be seen without Gauss sums. The field $K = \mathbb{Q}(\sqrt{d})$ has discriminant d, so $\zeta_K(s) = \zeta(s)L(s,\chi_d)$. Put $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$. The completed Dedekind zeta functions $\Lambda_K(s) = |d|^{s/2}\Gamma_{\mathbb{R}}(s)^{2-2\mathfrak{a}}\Gamma_{\mathbb{C}}(s)^{\mathfrak{a}}\zeta_K(s)$ and $\Lambda_{\mathbb{Q}}(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ are invariant under $s \mapsto 1-s$ by Hecke's theorem [13, Ch. VII, §5], and their quotient is $|d|^{s/2}\Gamma_{\mathbb{R}}(s+\mathfrak{a})L(s,\chi_d) = |d|^{-\mathfrak{a}/2}\Lambda(s)$. Therefore $\Lambda(\frac{1}{2}+t)$ is even in t. The order of Λ at $\frac{1}{2}$ equals m_d because the gamma factor is finite and nonzero there. □

2.4 Restricted cumulants

Definition 5. Let $f_1, \dots, f_n$ be even real functions whose Fourier transforms $h_i = \widehat{f_i}$ are profiles. Put

$$\Delta(f_1,\dots,f_n) = -\frac{(-2)^n}{2}\int_1^\infty (h_1^{\mathrm{o}} * \cdots * h_n^{\mathrm{o}})(u)\,\mathrm{d}u.$$

Define

$$\begin{aligned}
\mathcal{K}(f_1) &= h_1(0) - \int_0^1 h_1(u)\,\mathrm{d}u, \qquad \text{when } h_1 \text{ is continuous at } 0,\\
\mathcal{K}(f_1,f_2) &= 2\int_{\mathbb{R}} |u| h_1(u)h_2(u)\,\mathrm{d}u + \Delta(f_1,f_2),\\
\mathcal{K}(f_1,\dots,f_n) &= \Delta(f_1,\dots,f_n) \qquad (n \geq 3),
\end{aligned}$$

and, for $k \geq 1$,

$$\mathcal{M}(f_1,\dots,f_k) = \sum_{\pi}\prod_{B\in\pi}\mathcal{K}\big((f_i)_{i\in B}\big), \tag{3}$$

the sum running over all set partitions π of $\{1,\dots,k\}$.

Since h_1^{o} is odd, $\Delta(f_1) = \int_1^\infty h_1$, so that $\mathcal{K}(f_1) = h_1(0) - \frac{1}{2}f_1(0) + \Delta(f_1)$; here $f_1(0) = \int h_1 = 2\int_0^\infty h_1$. The quantities $\mathcal{K}$ are symmetric in their arguments. By Theorem 7 below, $\mathcal{M}$ gives the limiting mixed moments of $S_{f_1},\dots,S_{f_k}$ when the f_i are even Schwartz functions with $\widehat{f_i} \in C_c^\infty(\mathbb{R})$ and $\sum \mathrm{rad}(f_i) < 2$, so for such test functions the $\mathcal{K}$ are the limiting joint

cumulants. For a general profile the statistic S_f need not be defined: if $\widehat{f} = \mathbf{1}_{[-a,a]}$ then $f(x) = \sin(2\pi a x)/(\pi x)$, which does not satisfy $f(x) \ll (1+x^2)^{-1}$. For the rectangle and Legendre profiles used below, Theorem 22 passes to the limit by a separate smoothing argument. We call the $\mathcal{K}$ *restricted cumulants*. In the symplectic model they are the joint cumulants of linear statistics in the same range of supports; this is the content of [3, Theorem 7.2] written in cumulant form, and we do not use it.

Lemma 6. *Let $h_i = \widehat{f_i}$ be profiles with radii b_i.*

(a) If $b_1 + \cdots + b_n \le 1$ then $\Delta(f_1, \dots, f_n) = 0$.

(b) $|\Delta(f_1, \dots, f_n)| \le 2^{n-1} \prod_i \|h_i\|_1$.

(c) Let B, B' be disjoint subsets of $\{1, \dots, k\}$ with $\sum_{i \in B \cup B'} b_i \le 2$. Then $\Delta((f_i)_{i\in B})\,\Delta((f_i)_{i\in B'}) = 0$.

(d) If $h_i^{(j)} \to h_i$ in L^1 as $j \to \infty$, with $\sup_j \|h_i^{(j)}\|_\infty < \infty$ and all supports in a fixed compact set, then Δ and $\mathcal{K}$ of order $n \ge 2$ converge, and $\mathcal{K}(f_1^{(j)}) \to \mathcal{K}(f_1)$ if in addition $h_1^{(j)}(0) \to h_1(0)$.

Proof. (a) The convolution is supported in $[-\sum b_i, \sum b_i] \subseteq [-1, 1]$. (b) Young's inequality gives $\|h_1^{\circ} * \cdots * h_n^{\circ}\|_1 \le \prod \|h_i\|_1$. (c) If both factors are nonzero then by (a) $\sum_B b_i > 1$ and $\sum_{B'} b_i > 1$. (d) Telescoping and Young's inequality give convergence of the convolution in L^1, hence of Δ. The Gaussian term converges because $|u|$ is bounded on the common support, so $\int |u||h_1^{(j)} h_2^{(j)} - h_1 h_2| \ll \|h_1^{(j)} - h_1\|_1 \|h_2^{(j)}\|_\infty + \|h_1\|_\infty \|h_2^{(j)} - h_2\|_1$. The case $n = 1$ is clear. □

By Lemma 6(c), when $\sum_{i \le k} b_i \le 2$ each product in (3) contains at most one nonzero factor Δ.

3 Restricted correlations in prescribed patterns

Theorem 7. *Let $\mathcal{D}$ be one of the six classes and let $z = z(X) \ge 2$ satisfy $z = o(L)$. Let $f_1, \dots, f_k$ be even real Schwartz functions with $\widehat{f_i} \in C_c^\infty(\mathbb{R})$ and $\sum_i \operatorname{rad}(f_i) < 2$. Then*

$$\lim_{X \to \infty} \sup_{\upsilon} \left| \mathbb{E}_\upsilon[S_{f_1} \cdots S_{f_k}] - \mathcal{M}(f_1, \dots, f_k) \right| = 0,$$

the supremum running over all patterns υ of length z.

For $\mathcal{D} = \mathcal{D}(8,1)$ and the empty pattern, Theorem 7 is equivalent to Gao's n-level density theorem [4], in the form identified with the symplectic model in [3]. We give a self-contained proof because we need the five other classes and uniformity in the pattern. The proof follows the structure of Gao's argument, but evaluates the main terms directly and so does not need the comparison with random matrix theory. We use GRH for Dirichlet L-functions [1] in Lemmas 10 and 15 and through them in the rest of the section.

3.1 Notation for the proof

Fix the class $\mathcal{D} = \mathcal{D}(\kappa, \epsilon)$. For $\kappa \in \{1, 4\}$ the odd part satisfies $m \equiv b_0 \bmod 4$, where $b_0 = -\epsilon$ if $\kappa = 4$ and $b_0 = \epsilon$ if $\kappa = 1$; thus $\epsilon b_0 = -1$ for $\kappa = 4$ and $\epsilon b_0 = 1$ for $\kappa = 1$. Write $\psi_d(p) = \chi_d(p)$ for odd p; thus $\psi_d(p) = (\frac{2\epsilon m}{p})$ if $\kappa = 8$ and $\psi_d(p) = (\frac{\epsilon m}{p})$ otherwise. We extend this definition to every odd positive integer m' in place of m, obtaining the real Dirichlet character $\psi_{m'}$ (of modulus dividing $8m'$) on odd primes.

For a pattern v of length z put

$$Q = \prod_{2<p\le z} p, \quad r = \prod_{v_p=0} p, \quad s = Q/r, \quad t = \omega(s), \quad e_n = \prod_{p\mid n} v_p\,(n \mid s),$$

$$\mathcal{A} = \prod_{p\mid Q}\Big(1-\frac{1}{p}\Big)\prod_{p\nmid 2Q}\Big(1-\frac{1}{p^2}\Big).$$

Since $z = o(L)$ we have $\log Q \le (1+o(1))z = o(L)$, so Q, $2^t \le 2^{\pi(z)}$ and $\mathcal{A}^{-1} \ll \log(2z)$ are all $X^{o(1)}$, uniformly in v. All $o(1)$ terms below are uniform in v. For a profile F on $[0,\infty)$ (bounded, measurable, compactly supported) put

$$P_F(d) = \frac{1}{L}\sum_{p>2}\frac{\log p}{\sqrt{p}}\,F\Big(\frac{\log p}{L}\Big)\chi_d(p).$$

3.2 Counting

Lemma 8. *Uniformly in v,*

$$\mathfrak{M}_v(X) := \sum_{d\in\mathcal{D}_v} W\Big(\frac{m(d)}{X}\Big) = \theta_\kappa\,\widehat{W}(0)\,\frac{\mathcal{A}X}{r2^t}\big(1+O(X^{-1/2+o(1)})\big),$$

where $\theta_8 = \frac{1}{2}$ and $\theta_4 = \theta_1 = \frac{1}{4}$. In particular $\mathcal{D}_v \neq \varnothing$ for large X, and $\mathfrak{M}_v(X) \ge X^{1-o(1)}$.

Proof. Take $\kappa = 8$. The members of $\mathcal{D}_v$ have $m = rb$ with b odd, squarefree and prime to Q, and $\psi_d(p) = v_p$ for $p \mid s$. For such b the last condition has indicator $\prod_{p\mid s} \frac{1}{2}(1 + v_p\psi_{rb}(p)) = 2^{-t}\sum_{n\mid s} e_n\psi_{rb}(n)$, and $\psi_{rb}(n) = (\frac{2\epsilon r}{n})(\frac{b}{n})$. Hence

$$\mathfrak{M}_v(X) = 2^{-t}\sum_{n\mid s} e_n\Big(\frac{2\epsilon r}{n}\Big) \sum_{\substack{b \text{ squarefree}\\ (b,2Q)=1}} \Big(\frac{b}{n}\Big) W\Big(\frac{rb}{X}\Big).$$

Write $\mu^2(b) = \sum_{\alpha^2\mid b}\mu(\alpha)$ and detect $(b, 2Q) = 1$ by Möbius inversion. For a character ξ modulo q and $T \geq 1$ one has $\sum_c \xi(c)W(c/T) = \mathbf{1}_{\xi=\xi_0}\frac{\varphi(q)}{q}T\widehat{W}(0) + O(2^{\omega(q)})$ if ξ is principal, and $O(q^{1/2}\log q)$ otherwise, by partial summation and the Pólya–Vinogradov inequality. With $\alpha \leq (2X)^{1/2}$ and at most $2^{\omega(2Q)}$ Möbius terms, the term $n = 1$ is $\frac{X}{r}\widehat{W}(0)\prod_{p\mid 2Q}(1 - \frac{1}{p})\prod_{p\nmid 2Q}(1 - \frac{1}{p^2}) + O(X^{1/2+o(1)})$, and $\prod_{p\mid 2Q}(1 - \frac{1}{p})\prod_{p\nmid 2Q}(1 - \frac{1}{p^2}) = \mathcal{A}/2$. For $1 < n \mid s$ the character $b \mapsto (\frac{b}{n})$ is nonprincipal modulo $n \leq Q$, so these terms are $O(X^{1/2+o(1)})$. There are $2^t = X^{o(1)}$ of them, and $\mathcal{A}/(r2^t) \geq X^{-o(1)}$. For $\kappa \in \{1, 4\}$ the same argument applies with the extra condition $b \equiv b_0 r \bmod 4$, detected by the two characters modulo 4; this halves the main term.

The argument uses only that W is supported in $[1, 2]$ and has bounded variation: partial summation then gives the two character-sum estimates above with an extra factor $\mathrm{TV}(W)$. So the lemma holds, with the same proof, for every fixed weight W' of bounded variation supported in $[1, 2]$ in place of W, with $\int W'$ in place of $\widehat{W}(0)$; for example $W' = \mathbf{1}_{(1,2]}$ and $W' = \mathbf{1}_{(1,2]} - W$. □

Lemma 9. *Let $b \geq 1$ be an integer, all of whose prime factors exceed z, which is not a perfect square. Then uniformly in v, $\mathbb{E}_v[\chi_d(b)] \ll X^{-1/2+o(1)}\, b^{1/2+o(1)}$.*

Proof. Write $b = b^*c^2$ with $b^* > 1$ squarefree. Then $\chi_d(b) = \chi_d(b^*)\mathbf{1}_{(m,c)=1}$, and $\chi_d(b^*) = \psi_{rb'}(b^*)$ where $m = rb'$. Expanding the pattern as in Lemma 8 expresses $\mathfrak{M}_v\mathbb{E}_v[\chi_d(b)]$ as 2^t sums over squarefree b' prime to $2Qc$ of $W(rb'/X)$ times the character $b' \mapsto (\frac{b'}{nb^*})$, which is nonprincipal because $nb^* > 1$ is squarefree. The argument of Lemma 8 bounds each sum by $X^{1/2+o(1)}b^{1/2+o(1)}$, and Lemma 8 gives the denominator. □

3.3 The explicit formula and a priori bounds

For $0 < \beta \leq 1$ put

$$V_\beta(x) = \frac{\cosh(2\pi\beta) - \cos(2\pi\beta x)}{1 + x^2}.$$

Then $V_\beta(x) \geq (\cosh 2\pi\beta - 1)(1+x^2)^{-1} > 0$, and from $\int \mathrm{e}^{-2\pi i x\xi}(1+x^2)^{-1}\,\mathrm{d}x = \pi \mathrm{e}^{-2\pi|\xi|}$ one finds $\widehat{V}_\beta(\xi) = \pi\big(\cosh(2\pi\beta)\mathrm{e}^{-2\pi|\xi|} - \mathrm{e}^{-2\pi\beta}\cosh(2\pi\xi)\big)$ for $|\xi| \leq \beta$ and $\widehat{V}_\beta(\xi) = 0$ for $|\xi| \geq \beta$. So $\widehat{V}_\beta$ is a Lipschitz profile of radius β.

Lemma 10 (Explicit formula)**.** *Let f be even and real with $\widehat{f} \in C_c^\infty(\mathbb{R})$, or $f = V_\beta$. For $d \in \mathcal{D}$ with $X < m(d) \leq 2X$,*

$$S_f(d) = \widehat{f}(0) - \int_0^\infty \widehat{f}(u)\,\mathrm{d}u - 2P_{\widehat{f}}(d) + O_f\Big(\frac{\log L}{L}\Big).$$

Proof. For f with $\widehat{f} \in C_c^\infty$, Weil's explicit formula for the primitive character χ_d of conductor $|d|$ [6, Theorem 5.12] reads

$$S_f(d) = \frac{\widehat{f}(0)}{L}\log\frac{|d|}{\pi} + \frac{1}{L}\int_{\mathbb{R}} f(x)\,\Re\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{\mathfrak{a}}{2} + \frac{\pi i x}{L}\Big)\,\mathrm{d}x \\ - \frac{2}{L}\sum_{n\geq 1}\frac{\Lambda(n)\chi_d(n)}{\sqrt{n}}\,\widehat{f}\Big(\frac{\log n}{L}\Big),$$

with $\mathfrak{a} \in \{0,1\}$ as in Lemma 4. By GRH [1] every zero is $\frac{1}{2} + i\gamma$ with γ real, so the zero side is $S_f(d)$. Since $\log|d| = L + O(1)$ the first term is $\widehat{f}(0) + O(1/L)$, and the integral is $O_f(1/L)$. The terms $n = p^k$ with $k \geq 3$ contribute $O_f(1/L)$, the term $p = 2$ contributes $O_f(1/L)$, and the terms $n = p$ with $p > 2$ give $-2P_{\widehat{f}}(d)$. For $n = p^2$ we have $\chi_d(p)^2 = 1$ unless $p \mid d$. The primes dividing d contribute $O_f(L^{-1}\sum_{p\mid d}\frac{\log p}{p}) = O_f(\log L/L)$, and by the prime number theorem $\frac{2}{L}\sum_p \frac{\log p}{p}\widehat{f}(\frac{2\log p}{L}) = \int_0^\infty \widehat{f} + O_f(1/L)$. For $d = 8m > 0$ the resulting formula is Gao's [4, Lemma 2.3] (numbered as in the arXiv version), in which the gamma factor and the higher prime powers are already absorbed into an error term $O(\log\log X/\log X)$; it corroborates the reduction above, which we need in all six classes. For $f = V_\beta$ apply the formula to $V_{\beta,\eta}$ with $\widehat{V}_{\beta,\eta} = \widehat{V}_\beta * \rho_\eta$, ρ_η a smooth even approximate identity supported in $[-\eta,\eta]$, and let $\eta \to 0$: the prime side is a finite sum, and the zero side converges by dominated convergence since $|V_{\beta,\eta}(x)| \ll (1+x^2)^{-1}$ uniformly in η (two integrations by parts, $\widehat{V}_\beta$ being Lipschitz and piecewise smooth). The error term is uniform in η. □

Lemma 11 (Low frequencies)**.** *Let $a \geq 2$ be even and $\delta > 0$ with $4a\delta < 1$. There is C_a such that for every bounded profile F on $[0,\infty)$ with $F(u) = 0$ for all $u > 2\delta$, writing $M_F = \sup_{u\geq 0}|F(u)|$ for its pointwise supremum,*

$$\limsup_{X\to\infty}\ \sup_{v}\ \mathbb{E}_v\big[P_F^a\big] \leq C_a\big(M_F\delta\big)^a.$$

Proof. The primes $p \le z$ contribute at most $M_F L^{-1} \sum_{p \le z} \log p/\sqrt{p} \ll M_F \sqrt{z}/L = o(1)$ to P_F, pointwise. By Minkowski's inequality it suffices to treat $P_F^{>}$, the sum over $p > z$. Expanding,

$$\mathbb{E}_v\big[(P_F^{>})^a\big] = L^{-a} \sum_{z < p_1, \dots, p_a} \prod_i \frac{\log p_i}{\sqrt{p_i}} F\Big(\frac{\log p_i}{L}\Big) \mathbb{E}_v[\chi_d(p_1 \cdots p_a)].$$

Suppose first that $p_1 \cdots p_a$ is not a square. Since $p_1 \cdots p_a \le X^{2a\delta}$, Lemma 9 bounds the mean by $X^{-1/2+o(1)} \cdot X^{(1/2+o(1))2a\delta}$. The number of such tuples, weighted by $\prod \log p_i/\sqrt{p_i}$, is $\le X^{a\delta+o(1)}$, so these terms contribute $M_F^a X^{2a\delta-1/2+o(1)} = o(1)$. If $p_1 \cdots p_a$ is a square, every prime occurs an even number of times. Bounding the mean by 1 and grouping indices into blocks of equal primes, each block of size 2 contributes at most $M_F^2 L^{-2} \sum_{p \le X^{2\delta}} (\log p)^2/p \le M_F^2(2\delta^2 + O(L^{-1}))$ and each larger block $O(M_F^{|B|} L^{-|B|})$. □

Lemma 12 (Bounded moments)**.** *Let f be even and real with $\widehat{f} \in C_c^\infty$, and $k \ge 1$. Then $\mathbb{E}_v|S_f|^k$ and $\mathbb{E}_v|P_{\widehat{f}}|^k$ are bounded uniformly in v for large X.*

Proof. Choose an even $a \ge k$ and $\beta \in (0,1)$ with $2a\beta < 1$. Since $|f(x)| \ll_f (1+x^2)^{-1} \ll_\beta V_\beta(x)$ and the ordinates are real, $|S_f| \ll S_{V_\beta}$. By Lemma 10, $S_{V_\beta} = O(1) - 2P_{\widehat{V_\beta}}$, and Lemma 11 with $2\delta = \beta$ bounds $\mathbb{E}_v[P_{\widehat{V_\beta}}^a]$. Hölder's inequality gives the claim for S_f, and Lemma 10 then gives it for $P_{\widehat{f}}$. □

3.4 Reduction to distinct primes

For profiles $F_1, \dots, F_a$ on $[0,\infty)$ define the distinct-prime statistic and its predicted mean

$$\mathcal{P}^*(F_1, \dots, F_a)(d) = L^{-a} \sideset{}{^*}\sum_{p_1, \dots, p_a} \prod_{i=1}^{a} \frac{\log p_i}{\sqrt{p_i}} F_i\Big(\frac{\log p_i}{L}\Big) \chi_d(p_1 \cdots p_a),$$

$$\Xi(F_1, \dots, F_a) = -\frac{1}{2} \sum_{I \subseteq \{1,\dots,a\}} (-1)^{|I|} \int_{[0,\infty)^a} \mathbf{1}\big[\textstyle\sum_i \tau_i u_i > 1\big] \displaystyle\prod_i F_i(u_i)\, \mathrm{d}u,$$

where $\tau_i = -1$ for $i \in I$ and $\tau_i = 1$ for $i \notin I$, and $\sum^*$ runs over pairwise distinct odd primes; put $\mathcal{P}^*(\varnothing) = \Xi(\varnothing) = 1$. If h_i is the even extension of F_i, then substituting $y_i = \tau_i u_i$ shows

$$\begin{aligned} \Xi(F_1, \dots, F_a) &= -\frac{1}{2} \int_1^\infty (h_1^{\mathrm{o}} * \cdots * h_a^{\mathrm{o}})(x)\, \mathrm{d}x \\ &= (-2)^{-a} \Delta(f_1, \dots, f_a), \qquad \widehat{f_i} = h_i. \end{aligned} \tag{4}$$

We say that profiles $F_1, \dots, F_a$ are *admissible splines* if each F_i is a C^1 function on $[0,\infty)$ that is a polynomial on each interval between consecutive points of a finite set of knots, vanishes outside $[d_0, \sigma_i]$ for some $d_0 > 0$, with $\sum_i \sigma_i < 2$; and if they are in *general position*: no sum $\sum_{i \in I'} \tau_i \kappa_i$, with $\varnothing \neq I' \subseteq \{1, \dots, a\}$, signs τ_i, and κ_i a knot of F_i, equals 1. (The endpoints of the support count as knots.)

Proposition 13. *If $F_1, \dots, F_a$ ($a \geq 1$) are admissible splines, then $\sup_v |\mathbb{E}_v \mathcal{P}^*(F_1, \dots, F_a) - \Xi(F_1, \dots, F_a)| \to 0$.*

We prove Proposition 13 in Sections 3.5 and 3.6. First we deduce the moments of the prime sums.

Proposition 14. *Let $f_1, \dots, f_k$ be as in Theorem 7 and $F_i = \widehat{f_i}|_{[0,\infty)}$. Uniformly in v,*

$$\mathbb{E}_v\Big[\prod_{i=1}^{k} P_{F_i}\Big] = \sum_{\mu} \prod_{\{i,j\} \in \mu} \Big(\int_0^\infty u F_i(u) F_j(u)\, \mathrm{d}u \Big) \cdot \Xi\big((F_i)_{i \notin \cup \mu}\big) + o(1),$$

the sum running over all partial matchings μ of $\{1, \dots, k\}$ (sets of disjoint pairs, including $\mu = \varnothing$).

Proof. Let $b_i = \mathrm{rad}(f_i)$. Fix an even $a \geq k$ and $\delta > 0$ with $4a\delta < 1$, and a smooth $\chi_\delta : [0,\infty) \to [0,1]$ equal to 1 on $[0,\delta]$ and 0 on $[2\delta, \infty)$. Write $F_i = F_i^\flat + F_i^\sharp$ with $F_i^\flat = \chi_\delta F_i$. By Lemma 11, $\|P_{F_i^\flat}\|_{L^a(\mathbb{E}_v)} \ll \delta$, and by Lemma 12 $\|P_{F_i^\sharp}\|_{L^a(\mathbb{E}_v)} \ll 1$, in the limit superior uniformly in v. By Hölder's inequality,

$$\mathbb{E}_v\Big[\prod_i P_{F_i}\Big] = \mathbb{E}_v\Big[\prod_i P_{F_i^\sharp}\Big] + O(\delta) + o(1). \tag{5}$$

Next fix $\eta > 0$. Each $F_i^\sharp$ is smooth and supported in $[\delta, b_i]$. Choose C^1 piecewise cubic splines G_i (cubic Hermite interpolation of the values and first derivatives of $F_i^\sharp$ on a fine mesh) such that G_i is supported in $[\delta/2, b_i + \eta]$ with $\sum_i (b_i + \eta) < 2$, $\|F_i^\sharp - G_i\|_\infty + \|(F_i^\sharp - G_i)''\|_{L^1} \leq \eta$, and such that $G_1, \dots, G_k$ are in general position. The last condition excludes finitely many affine hyperplanes in the space of mesh nodes, so it holds after an arbitrarily small perturbation of the nodes. Let g_i be the even function with $\widehat{g_i}$ the even extension of $F_i^\sharp - G_i$. Since $F_i^\sharp - G_i$ vanishes near 0, two integrations by parts give $|g_i(x)| \ll \eta(1+x^2)^{-1}$, so $|S_{g_i}| \ll \eta S_{V_{1/(4a)}}$. Lemma 10 extends to g_i (approximate $\widehat{g_i}$ by smooth profiles as in its proof), and so $P_{F_i^\sharp} - P_{G_i} =$

$\frac{1}{2}(O(\eta) - S_{g_i}) + o(1)$ has $L^a(\mathbb{E}_v)$ norm $O(\eta) + o(1)$ by Lemma 12's argument. By Hölder's inequality again, $\mathbb{E}_v[\prod P_{F_i^\sharp}] = \mathbb{E}_v[\prod P_{G_i}] + O(\eta) + o(1)$.

We now expand $\mathbb{E}_v[\prod_i P_{G_i}]$. Write $w_i(p) = L^{-1}(\log p)p^{-1/2}G_i(\log p/L)$; it vanishes unless $p \geq X^{\delta/2}$, so all primes involved exceed z once X is large. Grouping the k-tuples of primes by the partition ρ of $\{1, \dots, k\}$ into classes of equal primes,

$$\prod_i P_{G_i}(d) = \sum_\rho \sideset{}{^*}\sum_{(q_B)_{B\in\rho}} \prod_{B\in\rho} \Big(\prod_{i\in B} w_i(q_B)\Big)\chi_d(q_B)^{|B|},$$

with the q_B pairwise distinct. Möbius inversion on the partition lattice removes the distinctness at the cost of merging blocks: the inner sum is a finite signed combination of products, over partitions ρ' coarser than ρ, of unrestricted sums $\sum_q \prod_{i\in B'} w_i(q)\chi_d(q)^{|B'|}$. For $|B'| = 1$ such a sum is $O(L^{O(1)})$ for every d by Lemma 15 below and partial summation, since χ_d is nonprincipal. For $|B'| = 2$ it is $O(L^2)$, and for $|B'| \geq 3$ it is $O(X^{-\delta/4}L^3)$. Hence every ρ with a block of size at least 3 contributes $O(X^{-\delta/4}L^{O(1)})$ pointwise. For ρ with blocks of size 1 and 2, the same inversion shows that removing the distinctness conditions that involve the pairs costs $O(X^{-\delta/4}L^{O(1)})$. For a pair $\{i, j\}$,

$$\begin{aligned}\sum_q w_i(q)w_j(q)\chi_d(q)^2 &= \frac{1}{L^2}\sum_q \frac{(\log q)^2}{q} G_iG_j\Big(\frac{\log q}{L}\Big) + O\Big(\frac{L^2}{X^{\delta/2}}\Big)\\ &= \int_0^\infty uG_i(u)G_j(u)\,\mathrm{d}u + O\Big(\frac{1}{L}\Big),\end{aligned}$$

because at most $2/\delta$ primes $q \geq X^{\delta/2}$ divide $m \leq 2X$, and by the prime number theorem. Therefore

$$\mathbb{E}_v\Big[\prod_i P_{G_i}\Big] = \sum_\mu \prod_{\{i,j\}\in\mu} \Big(\int_0^\infty uG_iG_j\Big)\,\mathbb{E}_v\big[\mathcal{P}^*((G_i)_{i\notin\cup\mu})\big] + o(1).$$

The splines $(G_i)_{i\in R}$ are admissible for every $R \subseteq \{1, \dots, k\}$, so Proposition 13 evaluates each mean as $\Xi((G_i)_{i\in R}) + o(1)$.

Finally let $\eta \to 0$ and then $\delta \to 0$. The integrals $\int uG_iG_j$ converge to $\int uF_iF_j$, and $\Xi((G_i)_{i\in R}) \to \Xi((F_i)_{i\in R})$ by (4) and Lemma 6(d), because $G_i \to F_i^\sharp \to F_i$ in L^1 with uniformly bounded norms and supports. The errors $O(\delta) + O(\eta)$ are uniform in v and in X. □

Lemma 15. *Let ξ be a nonprincipal Dirichlet character modulo q. For $y \geq 2$ we have $\sum_{p\leq y} \xi(p)\log p \ll y^{1/2}\log^2(qy)$. Hence $\sum_{N<p\leq N'} \xi(p)(\log p)/p \ll N^{-1/2}\log^2(qN)$ for $2 \leq N < N' \leq 2N$.*

Proof. This follows from GRH for the primitive character inducing ξ [1] by the classical argument [2, §20], [6, Theorem 5.15]; the primes dividing q contribute $O(\log q)$. The second bound follows by partial summation. □

3.5 Distinct primes: the classes $\mathcal{D}(8, \pm 1)$

Let $F_1, \dots, F_a$ be admissible splines, supported in $[d_0, \infty)$, with $\sigma = \sum \sigma_i < 2$, and take $\kappa = 8$. Throughout this subsection $p_1, \dots, p_a$ are distinct primes with $p_i \geq X^{d_0}$, $P = p_1 \cdots p_a \leq X^{\sigma}$ and $u_i = \log p_i / L$. Constants may depend on a, the splines, W and d_0.

Step 1: a pointwise bound. For every odd $m' \leq 2X$ the character $\psi_{m'}$ is nonprincipal, since $8\epsilon m'$ has odd 2-adic valuation and so is not a square. Hence, by Lemma 15, multivariable partial summation, and inclusion–exclusion over coincidences (as in the proof of Proposition 14),

$$L^{-a} \sum_{p_1, \dots, p_a}^{*} \prod_i \frac{\log p_i}{\sqrt{p_i}} F_i(u_i)\, \psi_{m'}(p_1 \cdots p_a) \ll L^{O(1)} \qquad \text{uniformly in } m'. \quad (6)$$

Step 2: opening the pattern and the sieve. The members of $\mathcal{D}_v$ have $m = rb'$ with b' odd, squarefree and prime to Q. For b' odd and prime to Q (squarefree or not) let $\mathbf{1}_v(rb') = 2^{-t} \sum_{n \mid s} e_n \psi_{rb'}(n)$ be the pattern indicator. Insert $\mu^2(b') = \sum_{\alpha^2 \mid b'} \mu(\alpha)$ and put $Z = 2^{2t}(1+L)^C$ with $C = C(a)$ large. The terms with $\alpha > Z$ involve $O(X/(rZ))$ values of $b' \leq 2X/r$, and by (6) each contributes $O(L^{O(1)})$. Relative to $\mathfrak{M}_v$ (Lemma 8) their total is $O(2^t L^{O(1)}/(\mathcal{A}Z)) = o(1)$. Since b' is odd and prime to Q, only α with $(\alpha, 2Q) = 1$ occur, and this condition is kept explicitly in the sums over α below. For $\alpha \leq Z$ we have $\alpha = X^{o(1)}$, and $\alpha < X^{d_0}$ for large X. Write $b' = \alpha^2 b''$. Since $\psi_{rb'}(n) = 0$ when $(b'', n) > 1$, the remaining condition $(b'', Q) = 1$ reduces to $(b'', Q/n) = 1$, which we detect by $\sum_{\ell \mid (b'', Q/n)} \mu(\ell)$. Writing $b'' = \ell u$ and $M = nP$, the product of characters is

$$\psi_{r\alpha^2\ell u}(n)\, \psi_{r\alpha^2\ell u}(P) = \Big(\frac{2\epsilon r \alpha^2 \ell u}{M}\Big) = \Big(\frac{2\epsilon r \ell}{M}\Big)\Big(\frac{u}{M}\Big),$$

because $(\alpha, M) = 1$. Thus, with $T = X/(r\ell\alpha^2)$,

$$\mathfrak{M}_v\,\mathbb{E}_v[\mathcal{P}^*] = \sum_{\substack{\alpha\le Z\\(\alpha,2Q)=1}} \mu(\alpha)\,2^{-t}\sum_{n\mid s} e_n \sum_{\ell\mid Q/n}\mu(\ell)\,L^{-a}\sideset{}{^*}\sum_{p_i}\prod_i \frac{\log p_i}{\sqrt{p_i}}F_i(u_i) \times\Big(\frac{2\epsilon r\ell}{M}\Big)\sum_{u\ \mathrm{odd}}\Big(\frac{u}{M}\Big)W\Big(\frac{u}{T}\Big)+o(\mathfrak{M}_v). \quad (7)$$

Step 3: Poisson summation. For odd squarefree $M > 1$ let $\varepsilon_M = 1$ if $M \equiv 1 \bmod 4$ and $\varepsilon_M = i$ if $M \equiv 3 \bmod 4$, so that $\sum_{c \bmod M}(\frac{c}{M})\mathrm{e}(kc/M) = \varepsilon_M\sqrt{M}(\frac{k}{M})$.

Lemma 16. *For odd squarefree $M > 1$ and $T > 0$,*

$$\Big(\frac{2}{M}\Big)\sum_{u\ \mathrm{odd}}\Big(\frac{u}{M}\Big)W\Big(\frac{u}{T}\Big) = \frac{T\varepsilon_M}{2\sqrt{M}}\sum_{k\in\mathbb{Z}}(-1)^k\Big(\frac{k}{M}\Big)\widehat{W}\Big(\frac{kT}{2M}\Big).$$

Proof. Poisson summation in residue classes modulo M gives, for every $U > 0$,

$$\sum_u\Big(\frac{u}{M}\Big)W\Big(\frac{u}{U}\Big) = \frac{U\varepsilon_M}{\sqrt{M}}\sum_k\Big(\frac{k}{M}\Big)\widehat{W}\Big(\frac{kU}{M}\Big).$$

Subtract the even u, using this with $U = T/2$ and $(\frac{2u}{M}) = (\frac{2}{M})(\frac{u}{M})$, and write $\widehat{W}(kT/M) = \widehat{W}(2k\cdot T/(2M))$ with $(\frac{k}{M}) = (\frac{2}{M})(\frac{2k}{M})$ in the first sum. □

Since $(\frac{2}{M})(\frac{2\epsilon r\ell}{M})(\frac{k}{M}) = (\frac{\epsilon kr\ell}{M})$, replacing k by ϵk turns the inner sum in (7), including its character, into

$$\frac{T\varepsilon_M}{2\sqrt{M}}\sum_{k\neq 0}(-1)^k\Big(\frac{kr\ell}{M}\Big)\widehat{W}\Big(\frac{\epsilon kT}{2M}\Big);$$

the term $k = 0$ vanishes as $M > 1$. Dividing by $\mathfrak{M}_v$, the prefactors combine with the prime weights into

$$\frac{\mu(\alpha)e_n\mu(\ell)}{\widehat{W}(0)\,\mathcal{A}\,\alpha^2\ell\sqrt{n}}\cdot\prod_i\frac{\log p_i}{p_iL}F_i(u_i)\cdot\varepsilon_M\sum_{k\neq0}(-1)^k\Big(\frac{kr\ell}{M}\Big)\widehat{W}\Big(\frac{\epsilon kT}{2M}\Big), \quad (8)$$

up to a factor $1 + O(X^{-1/2+o(1)})$.

Let $g = (r,\ell)$ and $s_0 = r\ell/g^2$, which is odd and squarefree. Call k *principal* if $k = \pm s_0 j^2$ with $j \ge 1$. Note that $\varepsilon_M = \frac{1+i}{2} + \frac{1-i}{2}(\frac{-1}{M})$ and $(\frac{-1}{M}) = (\frac{-1}{n})\prod_i(\frac{-1}{p_i})$.

Step 4: nonprincipal frequencies. If k is not principal then neither $kr\ell$ nor $-kr\ell$ is a square, so $p \mapsto (\frac{\pm kr\ell}{p})$ are nonprincipal characters of modulus dividing $4|k|r\ell$. Split each p_i into dyadic ranges $(N_i, 2N_i]$, put $N = \prod N_i$ and $H = r\ell\alpha^2 nN/X$. In each of the two terms of ε_M, the dependence on p_i is a nonprincipal character times $\frac{\log p_i}{p_i} F_i(u_i)$ times the smooth factor $\widehat{W}(\epsilon kT/(2nP))$. Since F_i is piecewise polynomial and $p\,\partial_p$ applied to $\widehat{W}(c/P)$ gives $-\frac{c}{P}\widehat{W}'(\frac{c}{P})$, all mixed derivatives $\prod_{i\in J}(p_i\partial_{p_i})$ of the weight are $O_A((1+|k|/H)^{-A})$. Multivariable partial summation with Lemma 15, and inclusion–exclusion for distinctness (a coincidence of $l \geq 2$ primes near N_i costs $O(N_i^{1-l}L^l) \leq O(N_i^{-l/2}L^l)$), bound the sum over the box by $O(N^{-1/2}L^{O(1)}\log^{2a}(|k|+2)(1+|k|/H)^{-A})$. Since $\sum_{k\neq 0}\log^{2a}(|k|+2)(1+|k|/H)^{-A} \ll HL^{O(1)}$ for $A \geq 2$, and after the prefactor in (8), the nonprincipal frequencies contribute

$$\ll \sum_{\alpha\leq Z}\sum_{n|s}\sum_{\ell|Q}\sum_{\text{boxes}} \frac{1}{\mathcal{A}\alpha^2\ell\sqrt{n}}\cdot\frac{r\ell\alpha^2 n\sqrt{N}}{X}L^{O(1)} \ll X^{\sigma/2-1+o(1)} = o(1).$$

Step 5: principal frequencies and the kernel. For $k = \pm s_0 j^2$ we have $(\frac{kr\ell}{M}) = (\frac{\pm 1}{M})\mathbf{1}_{(j,M)=1}$, because $kr\ell = \pm(r\ell/g)^2 j^2$ and $(r\ell, M) = 1$; also $(-1)^k = (-1)^j$. Put $y_j = s_0 j^2 T/(2M) = j^2 X/(2g^2\alpha^2 M)$ and

$$V(y) = \Re\big[(1+i)\widehat{W}(\epsilon y)\big], \qquad V^\star(y) = \Re\big[(1-i)\widehat{W}(\epsilon y)\big].$$

Checking $M \equiv 1$ and $M \equiv 3 \bmod 4$ separately (using $\widehat{W}(-y) = \overline{\widehat{W}(y)}$) gives

$$\varepsilon_M\Big[\widehat{W}(\epsilon y) + \Big(\frac{-1}{M}\Big)\widehat{W}(-\epsilon y)\Big] = V(y) + \Big(\frac{-1}{M}\Big)V^\star(y).$$

So the principal frequencies contribute

$$\sum_{\substack{j\geq 1\\(j,M)=1}} (-1)^j\Big[V(y_j) + \Big(\frac{-1}{M}\Big)V^\star(y_j)\Big]$$

in place of the k-sum in (8). Expand

$$\mathbf{1}_{(j,M)=1} = \sum_{h|n}\mu(h)\sum_{I\subseteq\{1,\dots,a\}}(-1)^{|I|}\mathbf{1}[hh' \mid j], \qquad h' = \prod_{i\in I} p_i,$$

and write $j = hh'l$. As hh' is odd, $(-1)^j = (-1)^l$, and

$$y_j = l^2\mathrm{e}^{-b}, \qquad b = L\Big(\sum_i \tau_i u_i - 1\Big) + w, \qquad w = \log\frac{2g^2\alpha^2 n}{h^2},$$

with τ_i as in the definition of Ξ. Note $|w| \ll \log Q + \log Z = o(L)$. Define, for an even Schwartz Φ,

$$B_\Phi(b) = \sum_{l\ge 1} (-1)^l \Phi\big(l^2 \mathrm{e}^{-b}\big).$$

Lemma 17. *Let Φ be a real Schwartz function on $[0,\infty)$. Then $B_\Phi(b) = -\frac{1}{2}\Phi(0) + O_A(\mathrm{e}^{-Ab})$ for $b \ge 0$, $B_\Phi(b) = O_A(\mathrm{e}^{Ab})$ for $b \le 0$, and for each $j \ge 1$ the derivative $B_\Phi^{(j)}(b)$ is $O_{A,j}(\mathrm{e}^{-A|b|})$.*

Proof. Let $\phi(x) = \Phi(x^2)$, an even Schwartz function, and $Y = \mathrm{e}^{b/2}$. Poisson summation gives $\sum_{l\in\mathbb{Z}} (-1)^l \phi(l/Y) = Y \sum_{k\in\mathbb{Z}} \widehat{\phi}(Y(k+\frac{1}{2})) = O_A(Y^{-A})$ for $Y \ge 1$, so $B_\Phi = \frac{1}{2}(\sum_{l\in\mathbb{Z}} -\phi(0)) = -\frac{1}{2}\Phi(0) + O_A(Y^{-A})$. For $Y \le 1$, $|B_\Phi| \le \sum_{l\ge 1} |\phi(l/Y)| \ll_A Y^A$. Finally $\frac{\mathrm{d}}{\mathrm{d}b} = \frac{1}{2} Y \frac{\mathrm{d}}{\mathrm{d}Y}$, and $Y\frac{\mathrm{d}}{\mathrm{d}Y}\phi(l/Y) = \phi_1(l/Y)$ with $\phi_1(x) = -x\phi'(x)$ even, Schwartz and vanishing at 0; iterate. □

We first dispose of $V^\star$. In that term the factor $(\frac{-1}{M})$ carries the non-principal character $(\frac{-1}{\cdot})$ on every p_i, and $a \ge 1$. For fixed α, n, ℓ, h, I the weight $B_{V^\star}(b)$ has bounded derivatives in each u_i after the factor L^{-1}, by Lemma 17. Partial summation in p_1 with Lemma 15, and trivial bounds for the other primes, give $O(X^{-d_0/2} L^{O(1)})$, which is $X^{-d_0/2+o(1)}$ after the sums over α, n, ℓ, h, I and the prefactors.

The remaining principal contribution, after (8), is

$$\sum_{\substack{\alpha\le Z\\(\alpha,2Q)=1}} \sum_{n\mid s} \sum_{\ell\mid Q/n} \sum_{h\mid n} \sum_{I} \frac{\mu(\alpha) e_n \mu(\ell)\mu(h)(-1)^{|I|}}{\widehat{W}(0)\mathcal{A}\alpha^2 \ell\sqrt{n}} \times \sum_{p_i}^{*} \prod_i \frac{\log p_i}{p_i L} F_i(u_i)\, B_V\Big(L\Big(\sum_i \tau_i u_i - 1\Big) + w\Big).$$

Step 6: integration over the primes. Since B_V is bounded, removing the distinctness of the p_i costs $O(L^{O(1)} \sum_{p\ge X^{d_0}} (\log p)^2/p^2) = O(X^{-d_0/2})$ per term. For the unrestricted sums, replace one prime at a time by an integral: by the prime number theorem with the error term from GRH and partial summation, $\sum_p \frac{\log p}{p} \Psi(\frac{\log p}{L}) = L\int_0^\infty \Psi(u)\,\mathrm{d}u + O(X^{-d_0/2} L^{O(1)} \mathrm{TV}(\Psi))$ for Ψ supported in $[d_0,\infty)$; the total variation of $u_i \mapsto F_i(u_i) B_V(L(\dots)+w)$ is

bounded uniformly by Lemma 17. The errors are $X^{-d_0/2+o(1)}$ in total. Thus the sum over primes becomes

$$\begin{aligned} J_I(w) &= \int_{[0,\infty)^a} \prod_i F_i(u_i)\, B_V\Big(L\Big(\sum_i \tau_i u_i - 1\Big) + w\Big)\,\mathrm{d}u \\ &= \int_{\mathbb{R}} G_I(x)\, B_V\big(L(x-1)+w\big)\,\mathrm{d}x, \end{aligned}$$

where G_I is the density of $\sum_i \tau_i u_i$, that is, the convolution of the functions $x \mapsto F_i(\tau_i x)$.

Lemma 18 (Polynomial transition)**.** *There are $\eta_0 > 0$, $D \geq 0$ and real numbers $c_0(L), \dots, c_D(L)$, bounded independently of L, such that uniformly for $|w| \leq \eta_0 L/2$,*

$$\begin{aligned} J_I(w) &= \sum_{j=0}^{D} c_j(L)\Big(\frac{w}{L}\Big)^j + O(\mathrm{e}^{-\eta_0 L/4}), \\ c_0(L) &= -\frac{\widehat{W}(0)}{2}\int_1^\infty G_I(x)\,\mathrm{d}x + O\Big(\frac{1}{L}\Big). \end{aligned}$$

Proof. Each function $x \mapsto F_i(\tau_i x)$ is a compactly supported piecewise polynomial whose knots are the points $\tau_i\kappa_i$, κ_i a knot of F_i. Expanding it by its derivative jumps, as in Section 7, writes it as a finite combination of the right-facing truncated powers $E_{c,j}(x) = (x-c)_+^j/j!$ with c among these knots. This is needed also when $\tau_i = -1$: a convolution of oppositely facing truncated powers need not converge. For right-facing truncated powers the convolution integral runs over a bounded interval, and the beta integral gives $E_{c,j} * E_{c',j'} = E_{c+c',j+j'+1}$. Hence G_I, the convolution of the compactly supported functions $x \mapsto F_i(\tau_i x)$, is a finite combination of truncated powers $E_{c,j}$ with c among the sums $\sum_i \tau_i\kappa_i$ of knots κ_i of F_i; so it is piecewise polynomial with knots among these sums. By general position 1 is not a knot, so G_I agrees with a polynomial g_I on $(1-\eta_0, 1+\eta_0)$ for some $\eta_0 > 0$. Write $B_V = -\frac{\widehat{W}(0)}{2}\mathbf{1}_{[0,\infty)} + R$, where $|R(b)| \ll \mathrm{e}^{-|b|}$ by Lemma 17 (note $V(0) = \widehat{W}(0)$). The step part contributes $-\frac{\widehat{W}(0)}{2}\big(\int_1^\infty G_I + \int_{1-w/L}^1 g_I\big)$, and the second integral is a polynomial in w/L without constant term. In $\int G_I(x)R(L(x-1)+w)\,\mathrm{d}x$, the range $|x-1| \geq \eta_0$ contributes $O(\mathrm{e}^{-\eta_0 L/2})$, and replacing G_I by g_I there costs $O(\mathrm{e}^{-\eta_0 L/4})$. Finally $\int_{\mathbb{R}} g_I(x)R(L(x-1)+w)\,\mathrm{d}x = \frac{1}{L}\int_{\mathbb{R}} g_I\big(1+\frac{\beta-w}{L}\big)R(\beta)\,\mathrm{d}\beta$ is a polynomial in w/L with coefficients $O(1/L)$, as R has finite moments of all orders. □

Step 7: summation over h, n, ℓ and α. For fixed α, n, ℓ, I put $w_0 = \log(2g^2\alpha^2 n)$, so that $w = w_0 - 2\log h$. By Lemma 18, $\sum_{h|n} \mu(h) J_I(w_0 - 2\log h)$ is, up to $O(2^{\omega(n)} \mathrm{e}^{-\eta_0 L/4})$, the product over $p \mid n$ of the difference operators $\mathrm{id} - (\text{shift by } -2\log p/L)$ applied to a polynomial of degree $\le D$ with bounded coefficients, evaluated at $w_0/L = o(1)$. It vanishes if $\omega(n) > D$ and is $O(\prod_{p|n} (\log p)/L)$ otherwise. Therefore the terms with $n > 1$ contribute

$$\ll \mathcal{A}^{-1} \sum_{\alpha} \frac{1}{\alpha^2} \sum_{\ell | Q} \frac{1}{\ell} \sum_{1 < n | s} \frac{1}{\sqrt{n}} \prod_{p | n} \frac{2\log p}{L}$$

$$\ll \log^2(2z) \Big(\exp\Big(\sum_{p \le z} \frac{2\log p}{\sqrt{p} L} \Big) - 1 \Big) \ll \frac{\sqrt{z} \log^2(2z)}{L} = o(1).$$

For $n = 1$ we have $h = 1$, $w = \log(2g^2\alpha^2)$, and Lemma 18 gives $J_I(w) = -\frac{\widehat{W}(0)}{2} \int_1^\infty G_I + O((1 + \log\alpha + \log g)/L)$. The error terms total $O(\mathcal{A}^{-1} \sum_\alpha \alpha^{-2} \sum_{\ell|Q} \ell^{-1}(1 + \log\alpha + \log\ell)/L) = O(\log^3(2z)/L) = o(1)$, using $\sum_{\ell|Q} \log\ell/\ell \ll \log^2(2z)$. Extending the sum over α to all α prime to $2Q$ costs $O(\mathcal{A}^{-1}/Z) = o(1)$, and

$$\sum_{(\alpha, 2Q) = 1} \frac{\mu(\alpha)}{\alpha^2} \sum_{\ell | Q} \frac{\mu(\ell)}{\ell} = \prod_{p \nmid 2Q} \Big(1 - \frac{1}{p^2}\Big) \prod_{p | Q} \Big(1 - \frac{1}{p}\Big) = \mathcal{A}.$$

Since $e_1 = 1$, the limit is $-\frac{1}{2} \sum_I (-1)^{|I|} \int_1^\infty G_I = \Xi(F_1, \dots, F_a)$. This proves Proposition 13 for $\kappa = 8$.

3.6 Distinct primes: the classes with $\kappa \in \{1, 4\}$

Now let $\kappa \in \{1, 4\}$, so $m \equiv b_0 \bmod 4$ and $\psi_{m'}(p) = (\frac{\epsilon m'}{p})$. We indicate the changes to Section 3.5.

Pointwise bound. For $\kappa = 4$ the integers $m' \equiv b_0 \bmod 4$ satisfy $\epsilon m' \equiv 3 \bmod 4$, and for $\epsilon = -1$ we have $\epsilon m' < 0$; in both cases $\epsilon m'$ is not a square and $\psi_{m'}$ is nonprincipal. For $(\kappa, \epsilon) = (1, 1)$ we delete all perfect squares m' from the ambient sum *before* inserting the squarefree sieve. This does not change the family, since a squarefree $m > X$ is not a square, and afterwards (6) holds for every remaining m'. The truncation of the sieve is then as before.

Poisson summation. Here u runs over $u \equiv b_1 \bmod 4$ with $b_1 \equiv b_0 r\ell \bmod 4$, and the character is $(\frac{\epsilon r\ell}{M})(\frac{u}{M})$. By the Chinese remainder theorem over $4M$, using that M is its own inverse modulo 4 and that 4 is a square modulo M,

$$\sum_{u \equiv b_1 (4)} \Big(\frac{u}{M}\Big) W\Big(\frac{u}{T}\Big) = \frac{T}{4M} \sum_{k \in \mathbb{Z}} i^{k b_1 M}\, \varepsilon_M \sqrt{M} \Big(\frac{k}{M}\Big) \widehat{W}\Big(\frac{kT}{4M}\Big).$$

The factor $\frac{1}{4}$ matches $\theta_\kappa = \frac{1}{4}$ in Lemma 8, so the prefactor (8) is unchanged. Pair k with $-k$ for $k > 0$ and put $x = kT/(4M)$. Checking $M \equiv 1, 3 \bmod 4$ separately, the coefficient of $(\frac{kr\ell}{M})\sqrt{M}$ is $A_k(x) + (\frac{-1}{M}) B_k(x)$ with

$$A_k(x) = \Re\big[(i^{kb_1} + i\epsilon\, i^{-kb_1}) \widehat{W}(x)\big], \qquad B_k(x) = \Re\big[(i^{kb_1} - i\epsilon\, i^{-kb_1}) \widehat{W}(x)\big].$$

The B_k terms carry the character $(\frac{-kr\ell}{p})$ on each p_i, nonprincipal because $-kr\ell < 0$. The A_k terms carry $(\frac{kr\ell}{p})$, nonprincipal unless $k = s_0 j^2$. All of these are bounded exactly as in Step 4. No splitting of the primes into residue classes is needed.

Principal frequencies. For $k = s_0 j^2$ we have $(\frac{kr\ell}{M}) = \mathbf{1}_{(j,M)=1}$ and $s_0 b_1 \equiv b_0 \bmod 4$, because $(r\ell/g)^2 \equiv 1 \bmod 8$. Hence $A_{s_0 j^2}(x) = c(j) V_\epsilon(x)$ with

$$V_\epsilon(x) = \Re\big[(1 + i\epsilon) \widehat{W}(x)\big], \qquad c(j) = 1 \ (j \text{ even}), \qquad c(j) = \epsilon b_0 \ (j \text{ odd}),$$

and $x = (j/y)^2$ with $y = 2g\alpha\sqrt{M/X}$. For $\kappa = 4$ we have $\epsilon b_0 = -1$, so $c(j) = (-1)^j$, and Steps 5–7 apply verbatim with V_ϵ in place of V (and no $V^\star$ term), with $w = \log(4g^2\alpha^2 n/h^2)$. Note $V_\epsilon(0) = \widehat{W}(0)$.

For $\kappa = 1$ we have $c(j) = 1$ and the sum over j is not alternating. Put $I_\epsilon = \int_0^\infty V_\epsilon(u^2)\,\mathrm{d}u$. Since $\int_0^\infty \mathrm{e}^{-2\pi i t u^2}\,\mathrm{d}u = \mathrm{e}^{-\pi i/4}/\sqrt{8t}$ for $t > 0$ (with Gaussian regularisation to justify exchanging integrals, t ranging over the support of W),

$$I_1 = \int_0^\infty W(u^2)\,\mathrm{d}u, \qquad I_{-1} = 0. \tag{9}$$

Using $\frac{\varphi(M)}{M} = \sum_{e \mid M} \frac{\mu(e)}{e}$ and the same expansion of $(j, M) = 1$ as before,

$$\sum_{\substack{j \geq 1 \\ (j,M)=1}} V_\epsilon\Big(\frac{j^2}{y^2}\Big) - y I_\epsilon \frac{\varphi(M)}{M} = \sum_{h \mid n} \sum_{I} \mu(h) (-1)^{|I|}\, \mathfrak{K}\Big(\frac{y}{hh'}\Big),$$

$$\mathfrak{K}(a) = \sum_{l \geq 1} V_\epsilon\Big(\frac{l^2}{a^2}\Big) - a I_\epsilon.$$

Poisson summation of the even Schwartz function $x \mapsto V_\epsilon(x^2)$ gives $\mathfrak{K}(a) = -\frac{1}{2}\widehat{W}(0) + O_A(a^{-A})$ for $a \geq 1$, and trivially $\mathfrak{K}(a) = O(a)$ for $a \leq 1$. The same holds for $(a\frac{\mathrm{d}}{\mathrm{d}a})^j\mathfrak{K}$, with main term 0: for $j = 1$ the function $x \mapsto -2x^2V_\epsilon'(x^2)$ has integral I_ϵ over $[0,\infty)$ by parts, which cancels $-aI_\epsilon$. Thus $b \mapsto \mathfrak{K}(\mathrm{e}^{b/2})$ satisfies the conclusions of Lemma 17 with exponent $\frac{1}{2}$ for $b \leq 0$, which suffices for Lemma 18 and Steps 6–7.

It remains to account for the linear term. If $\epsilon = -1$ it vanishes by (9). If $\epsilon = 1$, restoring the factor $T/(4\sqrt{M})$ shows that it contributes exactly

$$\frac{T}{4\sqrt{M}}\, yI_1 \frac{\varphi(M)}{M} = \frac{g\sqrt{X}}{2r\ell\alpha}\,\frac{\varphi(M)}{M}\int_0^\infty W(u^2)\,\mathrm{d}u$$

for fixed α, n, ℓ, P. The squares deleted from the ambient sum have $u = s_0 l^2$, which forces l odd and $s_0 \equiv b_1 \bmod 4$ (true, as $\epsilon b_0 = 1$), with $(\frac{s_0 l^2 r\ell}{M}) = \mathbf{1}_{(l,M)=1}$. Their contribution is

$$\sum_{(l,2M)=1} W\Big(\frac{(r\ell\alpha l/g)^2}{X}\Big) = \frac{g\sqrt{X}}{2r\ell\alpha}\,\frac{\varphi(M)}{M}\int_0^\infty W(u^2)\,\mathrm{d}u + O\big(2^{\omega(M)+1}\big),$$

by Möbius inversion over divisors of $2M$ and $|\sum_{l\geq 1}\Phi(l/U) - U\int_0^\infty \Phi| \leq \mathrm{TV}(\Phi)$. So the deleted squares cancel the linear term. The errors $O(2^{\omega(M)+1})$, restored with the prime weights and divided by $\mathfrak{M}_v \geq X^{1-o(1)}$, total $O(X^{-1+o(1)}\sum_{P\leq X^\sigma} P^{-1/2}) = O(X^{\sigma/2-1+o(1)})$, since each P has at most $a!$ ordered factorisations. Steps 6–7 now apply to the residual kernel and give $\Xi(F_1,\dots,F_a)$. This completes the proof of Proposition 13.

3.7 Proof of Theorem 7

Put $c_i = \widehat{f_i}(0) - \int_0^\infty \widehat{f_i}$ and $F_i = \widehat{f_i}|_{[0,\infty)}$. By Lemma 10, $S_{f_i} = c_i - 2P_{F_i} + E_i$ with $|E_i| \ll \log L/L$ uniformly, and by Lemma 12 and Hölder's inequality

$$\mathbb{E}_v\Big[\prod_i S_{f_i}\Big] = \sum_{J\subseteq\{1,\dots,k\}}\prod_{i\notin J} c_i\,(-2)^{|J|}\,\mathbb{E}_v\Big[\prod_{i\in J} P_{F_i}\Big] + o(1).$$

Proposition 14 evaluates each mean, and the theorem follows from the next lemma.

Lemma 19. *If $\sum_i \mathrm{rad}(f_i) \leq 2$, then*

$$\sum_J \prod_{i\notin J} c_i\,(-2)^{|J|}\sum_\mu \prod_{\{i,j\}\in\mu}\Big(\int_0^\infty uF_iF_j\Big)\,\Xi\big((F_i)_{i\in J\setminus\cup\mu}\big) = \mathcal{M}(f_1,\dots,f_k),$$

where μ runs over partial matchings of J.

Proof. Put $\Delta(\varnothing)=1$. We have $(-2)^2\int_0^\infty uF_iF_j = 2\int_{\mathbb{R}}|u|h_ih_j =: \gamma_{ij}$, and $(-2)^{|R|}\Xi(F_R)=\Delta(f_R)$ by (4). So the left side is the sum, over decompositions of $\{1,\dots,k\}$ into a set C, a matching μ and a remainder R, of $\prod_{i\in C}c_i\prod_\mu\gamma_{ij}\,\Delta(f_R)$. On the right, write $\mathcal{K}(f_i)=c_i+\Delta(f_i)$, $\mathcal{K}(f_i,f_j)=\gamma_{ij}+\Delta(f_i,f_j)$ and $\mathcal{K}=\Delta$ for larger blocks, and expand each product in (3). By Lemma 6(c) the terms with two factors Δ vanish, and blocks of size ≥ 3 have no non-Δ part. The remaining terms are indexed by a partition into singletons, pairs and at most one block R carrying Δ, which is the same index set with the same summands. □

4 Amplitudes and the model inner product

4.1 Model moments

Let $\mathfrak{S}$ be the polynomial algebra over $\mathbb{R}$ generated by formal symbols S_f, one for each even real function f whose Fourier transform is a profile continuous at 0. The *model expectation* is the linear functional $\mathbb{E}^{\mathrm{mod}}$ on $\mathfrak{S}$ with $\mathbb{E}^{\mathrm{mod}}[S_{f_1}\cdots S_{f_k}]=\mathcal{M}(f_1,\dots,f_k)$ and $\mathbb{E}^{\mathrm{mod}}[1]=1$. By Theorem 7, if every monomial of $\Pi\in\mathfrak{S}$ has total radius <2 and all its $\widehat{f}$ lie in C_c^∞, then $\mathbb{E}_v[\Pi(d)]\to\mathbb{E}^{\mathrm{mod}}[\Pi]$ uniformly in patterns of length $o(L)$, where $\Pi(d)$ is obtained by substituting $S_f(d)$ for S_f.

For $a>0$ let T_a be the even function with $\widehat{T_a}=\mathbf{1}_{[-a,a]}$, and in $\mathfrak{S}$ put $Z_a=S_{T_a}-\mathcal{K}(T_a)$. For $a,b\in(0,1]$ put

$$v(a,b)=\mathcal{K}(T_a,T_b)=2\min(a,b)^2-(a+b-1)_+^2; \tag{10}$$

the second term is $\Delta(T_a,T_b)$, since for $u>1$ the convolution $(\mathbf{1}^{\circ}_{[-a,a]}*\mathbf{1}^{\circ}_{[-b,b]})(u)=(a+b-u)_+$. For a list $A=(a_1,\dots,a_r)$ of widths define the *Wick product*

$$:Z_A:=\sum_{\mu}(-1)^{|\mu|}\prod_{\{i,j\}\in\mu}v(a_i,a_j)\prod_{i\notin\cup\mu}Z_{a_i}\in\mathfrak{S},$$

the sum running over partial matchings μ of $\{1,\dots,r\}$. Thus $:Z_\varnothing:=1$, $:Z_{(a)}:=Z_a$ and $:Z_{(a,b)}:=Z_aZ_b-v(a,b)$. Since centring removes singletons from the moment–cumulant expansion,

$$\mathbb{E}^{\mathrm{mod}}\Big[S_{f_0}\prod_{i=1}^{n}Z_{a_i}\Big]=\sum_{\pi}\prod_{B\in\pi}\mathcal{K}\big((g_i)_{i\in B}\big),\qquad g_0=f_0,\ g_i=T_{a_i}, \tag{11}$$

summed over the partitions π of $\{0,1,\dots,n\}$ with no singleton block $\{i\}$, $i\ge1$.

4.2 Amplitudes

For $s > 0$ and $c = (c_1, \dots, c_J) \in \mathbb{R}^J$ define the *Legendre profile*

$$\ell_{s,c}(u) = \mathbf{1}_{[-s,s]}(u) \sum_{j=1}^{J} c_j P_{2j}(u/s),$$

where P_n is the Legendre polynomial of degree n. Since $\int_{-1}^{1} P_{2j} = 0$ for $j \geq 1$, $\int \ell_{s,c} = 0$. Also $\|\ell_{s,c}\|_\infty \leq \|c\|_1$ and $\|\ell_{s,c}\|_1 \leq 2s\|c\|_1$, because $|P_n| \leq 1$ on $[-1, 1]$.

Definition 20. An *amplitude* is a finite list of *rows* (A_k, η_k), $1 \leq k \leq K$, where $A_k = (a_{k,1}, \dots, a_{k,r_k})$ is a list of widths in $(0, 1)$ ($r_k \geq 0$) and η_k is a profile with $\int \eta_k = 0$ and $\mathrm{rad}(\eta_k) \leq 1 - \sum A_k$, where $\sum A_k = \sum_j a_{k,j}$. We write $\check{\eta}_k$ for the even function with Fourier transform η_k, and

$$F(x) = \sum_{k=1}^{K} \check{\eta}_k(x) \, {:}Z_{A_k}{:}.$$

Let G be the even function with $\widehat{G} = H$, where

$$H(u) = \Big(\mathsf{A} \cos\frac{|u|}{2} + \Big(\frac{1}{4} - \mathsf{A}\Big) \sin\frac{|u|}{2}\Big)\mathbf{1}_{[-1,1]}(u), \qquad \mathsf{A} = \frac{1 + \cot(1/4)}{8}. \tag{12}$$

For rows k, l define the model inner products

$$\langle G, G\rangle = \mathcal{K}(G^2), \qquad \langle G, k\rangle = \mathbb{E}^{\mathrm{mod}}\big[S_{G\check{\eta}_k} {:}Z_{A_k}{:}\big],$$
$$\langle k, l\rangle = \mathbb{E}^{\mathrm{mod}}\big[S_{\check{\eta}_k\check{\eta}_l} {:}Z_{A_k}{::}Z_{A_l}{:}\big],$$

and extend bilinearly: $\langle G, F\rangle = \sum_k \langle G, k\rangle$, $\langle F, F\rangle = \sum_{k,l}\langle k, l\rangle$. These are formal versions of $\lim \mathbb{E}_v \sum_x G(x)F(x)$ and $\lim \mathbb{E}_v \sum_x F(x)^2$, where x runs over $\gamma L/(2\pi)$, $\gamma \in \Gamma_d$; the products $G\check{\eta}_k$ and $\check{\eta}_k\check{\eta}_l$ have Fourier transforms $H * \eta_k$ and $\eta_k * \eta_l$, which are continuous profiles.

4.3 The one-level extremal function

Lemma 21. $\int H = G(0) = 1$. *For every profile ρ with $\mathrm{rad}(\rho) \leq 1$, writing $\check{\rho}$ for the even function with $\widehat{\check{\rho}} = \rho$,*

$$\mathcal{K}(G\check{\rho}) = (H * \rho)(0) - \int_0^1 (H * \rho)(u)\,\mathrm{d}u = \lambda \int \rho,$$
$$\lambda = H(0) - \frac{1}{2} = \frac{\cot(1/4) - 3}{8}.$$

In particular $\langle G, G\rangle = \lambda$, and $\langle G, k\rangle = 0$ for every row with $r_k = 0$.

Proof. Put $\theta = \frac{1}{4}$, so $\mathsf{A} = (\cos\theta + \sin\theta)/(8\sin\theta)$. Using $\sin 2\theta + \cos 2\theta - 1 = 2\sin\theta(\cos\theta - \sin\theta)$ and $\cos 2\theta = (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$,

$$\int_{-1}^{1} H = 4\mathsf{A}\sin\tfrac{1}{2} + (1-4\mathsf{A})\big(1-\cos\tfrac{1}{2}\big) = 1 - \cos\tfrac{1}{2} + 4\mathsf{A}\big(\sin\tfrac{1}{2} + \cos\tfrac{1}{2} - 1\big) = 1.$$

Next, for $0 < u < 1$ we claim $H'(u) = -\frac{1}{2}H(1-u)$. Expanding $\cos(\frac{1}{2} - \frac{u}{2})$ and $\sin(\frac{1}{2} - \frac{u}{2})$, the claim is equivalent to the two identities $4\mathsf{A}(\sin\frac{1}{2} + \cos\frac{1}{2} - 1) = \cos\frac{1}{2}$ and $4\mathsf{A}(1 + \sin\frac{1}{2} - \cos\frac{1}{2}) = 1 + \sin\frac{1}{2}$; the first was just used, and the second is equivalent to it since $\cos\frac{1}{2}(1 + \sin\frac{1}{2} - \cos\frac{1}{2}) = (1 + \sin\frac{1}{2})(\sin\frac{1}{2} + \cos\frac{1}{2} - 1)$.

Since H and ρ are even, $\int_0^1 (H * \rho) = \frac{1}{2}\int_{-1}^{1}(H * \rho)$, and this equals $\frac{1}{2}\int \rho(s)\int_{s-1}^{s+1} H(t)\,\mathrm{d}t\,\mathrm{d}s$. Hence

$$(H*\rho)(0) - \int_0^1 (H*\rho) = \int \rho(s)\Psi(s)\,\mathrm{d}s, \qquad \Psi(s) = H(s) - \frac{1}{2}\int_{s-1}^{s+1} H(t)\,\mathrm{d}t.$$

Ψ is even. For $0 < s < 1$, $\Psi'(s) = H'(s) - \frac{1}{2}(H(s+1) - H(s-1)) = H'(s) + \frac{1}{2}H(1-s) = 0$, and $\Psi(0) = H(0) - \frac{1}{2}\int H = \mathsf{A} - \frac{1}{2} = \lambda$. So $\Psi = \lambda$ on $[-1,1]$, which proves the formula. Taking $\rho = H$ gives $\langle G, G\rangle = \lambda\int H = \lambda$; taking $\rho = \eta_k$ with $\int \eta_k = 0$ gives the last claim. □

Note that $1 - \lambda/2 = (19 - \cot\frac{1}{4})/16$.

5 The variational bound

Theorem 22. *Let F be an amplitude, and put $C = \langle G, F\rangle$ and $D = \langle F, F\rangle$. Let $\mathcal{D}$ be one of the six classes and $z = o(L)$. Then*

$$\limsup_{X\to\infty} \sup_{v} \frac{\sum_{d\in\mathcal{D}_v} W(m(d)/X)\,\mathbf{1}[L(\frac{1}{2},\chi_d) = 0]}{\sum_{d\in\mathcal{D}_v} W(m(d)/X)} \;\le\; \frac{1}{2}\big(\lambda - 2C + D\big),$$

the supremum running over patterns v of length z. If $D > 0$, the right side may be replaced by $\frac{1}{2}(\lambda - C^2/D)$. Equivalently, the lower limit of the weighted nonvanishing proportion is at least $(19 - \cot\frac{1}{4})/16 + C - D/2$, and at least $(19 - \cot\frac{1}{4})/16 + C^2/(2D)$ if $D > 0$, uniformly in v.

Proof. Smoothing. Fix an even nonnegative $\rho \in C_c^\infty([-1,1])$ with $\int\rho = 1$. For $\lambda' \in (\frac{1}{2}, 1)$ and $\eta' > 0$ define, for a profile h,

$$\mathfrak{S}h = \big(\lambda'^{-1}h(\cdot/\lambda')\big) * \rho_{\eta'}, \qquad \rho_{\eta'}(u) = \eta'^{-1}\rho(u/\eta').$$

Then $\mathfrak{S}h \in C_c^\infty$, $\int \mathfrak{S}h = \int h$, $\operatorname{rad}(\mathfrak{S}h) \le \lambda' \operatorname{rad}(h) + \eta'$, $\|\mathfrak{S}h\|_\infty \le 2\|h\|_\infty$, and $\mathfrak{S}h \to h$ in L^1 as $(\lambda', \eta') \to (1, 0)$, with supports in a fixed compact set.

Apply $\mathfrak{S}$ to H, to each η_k and to each $\mathbf{1}_{[-a,a]}$, and put $\eta_k^\natural = \mathfrak{S}\eta_k$. Let $G^\natural$, $\check{\eta}_k^\natural$ and $T_a^\natural$ be the inverse Fourier transforms of the smoothed profiles: the even functions with $\widehat{G^\natural} = \mathfrak{S}H$, $\widehat{\check{\eta}_k^\natural} = \eta_k^\natural$ and $\widehat{T_a^\natural} = \mathfrak{S}\mathbf{1}_{[-a,a]}$. They are Schwartz functions, since the smoothed profiles lie in C_c^∞. Let $Z_a^\natural = S_{T_a^\natural} - \mathcal{K}(T_a^\natural)$ and form the Wick products $:Z_A^\natural:$ with the constants $v^\natural(a,b) = \mathcal{K}(T_a^\natural, T_b^\natural)$. For $z_0 \in \mathbb{R}$ and $d \in \mathcal{D}_v$ define

$$\mathcal{A}_d(x) = G^\natural(x) - z_0 \sum_k \check{\eta}_k^\natural(x)\, :Z_{A_k}^\natural:(d).$$

Since $\int \mathfrak{S}h = \int h$, we have $G^\natural(0) = G(0) = 1$ and $\check{\eta}_k^\natural(0) = 0$, so $\mathcal{A}_d(0) = 1$ for every d.

Pointwise inequality. By Lemma 4, if $L(\frac{1}{2}, \chi_d) = 0$ then Γ_d contains 0 with multiplicity $m_d \ge 2$, so

$$2 \cdot \mathbf{1}[L(\tfrac{1}{2}, \chi_d) = 0] \le \sum_{\gamma \in \Gamma_d} \mathcal{A}_d\Big(\frac{\gamma L}{2\pi}\Big)^2 =: N(d).$$

Expanding the square,

$$N = S_{(G^\natural)^2} - 2z_0 \sum_k S_{G^\natural \check{\eta}_k^\natural} :Z_{A_k}^\natural: + z_0^2 \sum_{k,l} S_{\check{\eta}_k^\natural \check{\eta}_l^\natural} :Z_{A_k}^\natural ::Z_{A_l}^\natural:,$$

a polynomial in linear statistics of functions with C_c^∞ Fourier transforms. Let $r = \max_k r_k$. A monomial of the last sum is $S_{\check{\eta}_k^\natural \check{\eta}_l^\natural}$ times at most $r_k + r_l \le 2r$ statistics $S_{T_a^\natural}$ with a from A_k and A_l; its total radius is at most

$$\lambda'\Big(2 - \sum A_k - \sum A_l\Big) + 2\eta' + \lambda'\Big(\sum A_k + \sum A_l\Big) + (r_k + r_l)\eta' \le 2\lambda' + (2r+2)\eta'.$$

The other sums are similar. Choose $(2r+2)\eta' < 2(1-\lambda')$; then every monomial has total radius < 2.

Limit in X. By Theorem 7, $\mathbb{E}_v[N] \to \mathbb{E}^{\mathrm{mod}}[N] =: \mathcal{Q}^\natural(z_0)$ uniformly in v, so the left side of the theorem is at most $\frac{1}{2}\mathcal{Q}^\natural(z_0)$.

Removing the smoothing. $\mathcal{Q}^\natural(z_0)$ is a finite combination of products of restricted cumulants of the functions $G^\natural$, $\check{\eta}_k^\natural$, $T_a^\natural$ and their pairwise products, with coefficients that are products of the constants $v^\natural$. The Fourier transforms

involved are $\mathfrak{S}h$ for the elementary profiles h, and convolutions $\mathfrak{S}h_1 * \mathfrak{S}h_2$ for the products; the latter converge uniformly, by Young's inequality, and in L^1. By Lemma 6(d), as $(\lambda', \eta') \to (1, 0)$ every cumulant converges to the corresponding cumulant of the unsmoothed functions (means only occur for the products, whose transforms converge at 0, and for the T_a, whose centring constants cancel in (11)). Hence $\mathcal{Q}^\natural(z_0) \to \mathcal{Q}(z_0) = \langle G, G\rangle - 2z_0C + z_0^2 D$, and the left side of the theorem is at most $\frac{1}{2}\mathcal{Q}(z_0)$ for every z_0. Use $\langle G, G\rangle = \lambda$ from Lemma 21, and take $z_0 = 1$, or $z_0 = C/D$ if $D > 0$. □

Remark. In the same way $\langle F, F\rangle = \lim\lim \mathbb{E}_v \sum_x F^\natural(x)^2 \geq 0$ for every amplitude, and $\langle F, F\rangle\langle G, G\rangle \geq \langle G, F\rangle^2$.

We shall build the amplitude in stages, and to avoid recomputing old norms we use the following trivial consequence of Theorem 22's quadratic form.

Lemma 23 (Chaining). *Let $F = F' + F''$ be amplitudes with $0 < \langle G, F'\rangle \leq U$ and $\langle G, F'\rangle^2 \geq 2g\langle F', F'\rangle$ for some $g > 0$. Then $\langle F, F\rangle \leq U^2/(2g) + 2\langle F', F''\rangle + \langle F'', F''\rangle$.*

6 Wick-ordered moments

Fix lists $A = (a_1, \ldots, a_r)$ and $B = (b_1, \ldots, b_s)$ of widths in $(0, 1)$ with $\sum A \leq 1$ and $\sum B \leq 1$, and an even function ϕ whose Fourier transform is a profile continuous at 0 with $\mathrm{rad}(\phi) + \sum A + \sum B \leq 2$. We label the variables by 0 (the *marked* variable S_ϕ) and by the entries of A and B. For a set β of labels write $\mathcal{K}(\beta)$ for the restricted cumulant of the corresponding functions (ϕ for 0, T_a for an entry a).

Lemma 24. $\mathbb{E}^{\mathrm{mod}}[S_\phi{:}Z_A{::}Z_B{:}] = \sum_\pi \prod_{\beta\in\pi} \mathcal{K}(\beta)$, *summed over the partitions π of the labels in which every block not containing 0 meets both A and B.*

Proof. Expand both Wick products and apply (11). A term is indexed by matchings μ_A of A, μ_B of B and a partition π' of the remaining labels without unmarked singletons, with weight $(-1)^{|\mu_A|+|\mu_B|}\prod_{\mu_A\cup\mu_B} v \cdot \prod_{\pi'} \mathcal{K}$. Let $\pi = \pi' \cup \mu_A \cup \mu_B$. For a pair $\{i, j\}$ inside A, $v(a_i, a_j) = \mathcal{K}(T_{a_i}, T_{a_j})$. Hence, for a fixed partition π without unmarked singletons, the terms producing it are indexed by subsets of the set $\mathcal{I}(\pi)$ of its blocks that are pairs inside A or inside B, and they sum to $(1-1)^{|\mathcal{I}(\pi)|}\prod_\pi \mathcal{K}$. So only partitions with no such pairs survive. An unmarked block of size ≥ 3 inside A has $\mathcal{K} = \Delta = 0$ by Lemma 6(a), since $\sum A \leq 1$; similarly for B. □

Lemma 25 (Contraction formula)**.** *Write $\Delta(S)$ for $\Delta((T_a)_{a\in S})$ when $|S|\geq 2$ and $\Delta(S)=0$ when $|S|\leq 1$, and $\mathcal{K}(\phi;S)$ for the cumulant of ϕ together with $(T_a)_{a\in S}$. Then*

$$\mathbb{E}^{\mathrm{mod}}[S_\phi{:}Z_A{::}Z_B{:}] = \sum_P \gamma(P)\Big[\mathcal{K}(\phi;S)+\Delta(S)\,\mathcal{K}(\phi;\varnothing)+\sum_{i\in S}\Delta(S\setminus\{i\})\,\mathcal{K}(\phi;\{i\})\Big],$$

where P runs over the partial matchings between A and B (each pair has one entry from each list), $\gamma(P)=\prod_{(a,b)\in P}2\min(a,b)^2$, and S is the list of entries of A and B not covered by P.

Proof. In Lemma 24 write the cumulant of each unmarked block β as $\gamma(\beta)+\Delta(\beta)$, where $\gamma(\beta)=2\min(a,b)^2$ if $\beta=\{a,b\}$ is a pair and $\gamma(\beta)=0$ otherwise, and expand. A term with two unmarked factors $\Delta(\beta)\Delta(\beta')\neq 0$ would need $\sum_\beta>1$ and $\sum_{\beta'}>1$, contradicting $\sum A+\sum B\leq 2$. If no unmarked factor Δ occurs, the unmarked blocks are pairs, each meeting both lists, so they form a cross-matching P, and the marked block is $\{0\}\cup S$; this gives $\gamma(P)\mathcal{K}(\phi;S)$. If exactly one unmarked block β contributes $\Delta(\beta)\neq 0$, then $\sum_\beta>1$, the other unmarked blocks form a cross-matching P with $\beta\subseteq S$, and the marked block $\{0\}\cup S'$, $S'=S\setminus\beta$, has total radius $\mathrm{rad}(\phi)+\sum S'<1$. If $|S'|\geq 2$ its cumulant is $\Delta=0$ by Lemma 6(a). So $S'=\varnothing$ or $S'=\{i\}$, giving the last two terms; blocks β of size one do not occur, which is the convention $\Delta(S)=0$ for $|S|\leq 1$. Conversely each term on the right arises once. □

Corollary 26. *For a row (A,η) of an amplitude with $r\geq 1$, $\langle G,(A,\eta)\rangle=\mathcal{K}(G\check{\eta};(T_a)_{a\in A})$, the restricted cumulant of order $r+1$.*

Proof. Apply Lemma 25 with $\phi=G\check{\eta}$ and $B=\varnothing$: then $P=\varnothing$, $S=A$, and $\Delta(A)=\Delta(A\setminus\{i\})=0$ since $\sum A\leq 1$. The radius condition is $\mathrm{rad}(H*\eta)+\sum A\leq 1+(1-\sum A)+\sum A=2$. □

For two rows k,l of an amplitude, Lemma 25 with $\phi=\check{\eta}_k\check{\eta}_l$, $A=A_k$, $B=A_l$ evaluates $\langle k,l\rangle$. The radius condition holds because $\mathrm{rad}(\eta_k*\eta_l)\leq \mathrm{rad}(\eta_k)+\mathrm{rad}(\eta_l)\leq 2-\sum A_k-\sum A_l$ by Definition 20. For the rows of Section 8, whose profiles have radius exactly $1-\sum A$, it holds with equality. When A_k and A_l contain repeated widths, the matchings may be grouped by multiplicity: if A consists of j_1 copies of a and B of j_2 copies of b, there are $\binom{j_1}{j}\binom{j_2}{j}j!$ matchings of size j.

7 Exact evaluation

All profiles in our amplitude are piecewise polynomial with rational knots and rational coefficients, except H, which we approximate. We now reduce every restricted cumulant needed to finitely many rational operations.

7.1 Truncated powers

For $c \in \mathbb{Q}$ and $j \geq 0$ let $E_{c,j}(u) = (u-c)_+^j/j!$. A compactly supported piecewise polynomial h is the finite sum $h = \sum_c \sum_j \big(h^{(j)}(c^+) - h^{(j)}(c^-)\big) E_{c,j}$ over its knots c. We have

$$E_{c,j} * E_{c',j'} = E_{c+c',\, j+j'+1}, \qquad \int_{-\infty}^{x} E_{c,j} = E_{c,j+1}(x),$$

the first by the beta integral. Hence for piecewise polynomial profiles h_1, h_2 with rational data, all lower primitives $B_k(x) = \int_{-\infty}^{x} \int \cdots \int (h_1 * h_2)$ (k-fold) are piecewise polynomial with rational coefficients, and take rational values at rational points.

Lemma 27. *Let ϕ be even with $\widehat{\phi} = h$ a profile continuous at 0, and $0 < t \leq 1$. Then*

$$\mathcal{K}(\phi) = h(0) - \int_0^1 h,$$
$$\mathcal{K}(\phi, T_t) = 4\int_0^t u h(u)\,\mathrm{d}u - 2\int_{1-t}^{1+t} \big(t - |1-u|\big) h(u)\,\mathrm{d}u.$$

Proof. The first is the definition. For the second, the Gaussian term is $2\int |u| h \mathbf{1}_{[-t,t]} = 4\int_0^t uh$. For $x > 1 \geq t$ and $|y| < t$ we have $x - y > 0$, so $(h^{\mathrm{o}} * \mathbf{1}^{\mathrm{o}}_{[-t,t]})(x) = \int_0^t (h(x-y) - h(x+y))\,\mathrm{d}y$, and integrating over $x > 1$ gives $\int_0^t \int_{1-y}^{1+y} h = \int_{1-t}^{1+t} (t - |1-u|) h(u)\,\mathrm{d}u$. Multiply by $-\frac{(-2)^2}{2} = -2$. $\square$

Lemma 28. *Let ϕ be as above, $n \geq 3$, and $a_1, \ldots, a_{n-1} \in (0,1]$. Let B_k be the k-fold lower primitive of $h = \widehat{\phi}$ and define*

$$A_n(x) = \begin{cases} -B_n(x), & x < 0, \\ B_n(x) - 2\sum_{k=0}^{n-1} \frac{x^k}{k!} B_{n-k}(0), & x \geq 0. \end{cases}$$

Then, with $(\mathsf{E}_a \Psi)(x) = \Psi(x+a)$,

$$\mathcal{K}(\phi, T_{a_1}, \ldots, T_{a_{n-1}}) = \frac{(-2)^n}{2} \Big[\prod_{j=1}^{n-1} \big(2 - \mathsf{E}_{a_j} - \mathsf{E}_{-a_j}\big) A_n \Big](1).$$

Proof. A_n is the n-fold lower primitive of h°: for $n = 1$ this is $\int_{-\infty}^{x} h^{\circ} = -B_1(x)$ for $x < 0$ and $-B_1(0) + (B_1(x) - B_1(0))$ for $x \geq 0$, and the general case follows by induction. The distributional derivative of $\mathbf{1}^{\circ}_{[-a,a]}$ is $2\delta_0 - \delta_a - \delta_{-a}$, so convolving with $\mathbf{1}^{\circ}_{[-a,a]}$ is the operator $2 - \mathsf{E}_a - \mathsf{E}_{-a}$ applied to the lower primitive. Hence $\Pi := \prod_j (2 - \mathsf{E}_{a_j} - \mathsf{E}_{-a_j}) A_n$ is the lower primitive of $h^{\circ} * \mathbf{1}^{\circ}_{[-a_1,a_1]} * \cdots$. This convolution has compact support and total integral $\int h^{\circ} \cdot \prod \int \mathbf{1}^{\circ} = 0$, so $\int_1^{\infty} = -\Pi(1)$, and the definition of Δ gives the claim. □

7.2 The extremal function

Lemma 29. *Let* $\mathsf{A}^* = 61453967058074 \cdot 10^{-14}$ *and let* H^* *be the even function, vanishing outside* $[-1,1]$*, that equals on* $[0,1]$ *the Taylor polynomial of degree* 12 *at* 0 *of* $\mathsf{A}^* \cos(u/2) + (\frac{1}{4} - \mathsf{A}^*)\sin(u/2)$*. Then* $|H - H^*| \leq 10^{-12}$ *everywhere, hence* $\|H - H^*\|_1 \leq 2 \cdot 10^{-12}$*.*

Proof. The alternating Taylor series of $\cos$ and $\sin$ at $\frac{1}{4}$, truncated after degrees $12, 14$ and $13, 15$, give rational enclosures of $\cot\frac{1}{4}$ which imply $|\mathsf{A} - \mathsf{A}^*| < 5 \cdot 10^{-15}$. The Taylor remainders of $\cos(u/2)$ and $\sin(u/2)$ for $|u| \leq 1$ are at most $2^{-14}/14!$ and $2^{-13}/13!$, and $|\mathsf{A}|, |\frac{1}{4} - \mathsf{A}| < 1$. Altogether $|H - H^*| \leq 2^{-14}/14! + 2^{-13}/13! + 10^{-14} < 10^{-12}$. □

Write G^* for the even function with $\widehat{G^*} = H^*$.

Lemma 30. *Let* (A, η) *be a row with* $r = |A| \geq 1$ *and* $\eta = \ell_{s,c}$*,* $s = 1 - \sum A$*. Then*

$$\left|\mathcal{K}(G\check{\eta}; T_A) - \mathcal{K}(G^*\check{\eta}; T_A)\right| \leq \begin{cases} 8a^2 s\|c\|_1 \cdot 10^{-12}, & A = (a), \\ 2^r \cdot 2 \cdot 10^{-12} \cdot 2s\|c\|_1 \prod_{a \in A}(2a), & r \geq 2. \end{cases}$$

Proof. The difference is the cumulant with $h = (H - H^*) * \eta$. For $r = 1$, Lemma 27 bounds it by $(2a^2 + 2a^2)\|h\|_\infty$, and $\|h\|_\infty \leq \|H - H^*\|_\infty \|\eta\|_1 \leq 10^{-12} \cdot 2s\|c\|_1$. For $r \geq 2$ use Lemma 6(b) with $n = r + 1$ and $\|h\|_1 \leq \|H - H^*\|_1 \|\eta\|_1$. □

Lemma 31. *We have* $\cot\frac{1}{4} < 3.916317365$*, and consequently* $p_0 := (19 - 3.916317365)/16 < (19 - \cot\frac{1}{4})/16$*.*

Proof. At $z = \frac{1}{4}$, the alternating partial sums $\sum_{k \leq 6} (-1)^k z^{2k}/(2k)!$ and $\sum_{k \leq 7} (-1)^k z^{2k+1}/(2k+1)!$ are an upper bound for $\cos z$ and a lower bound for $\sin z > 0$; their quotient is a rational number below 3.916317365. □

8 The amplitude and the certificate: proof of Theorem 1

8.1 The rows

The amplitude is assembled in eight stages from ten families of rows (R0)–(R9). Every profile is a rational multiple of a Legendre profile $\ell_{s,c}$ with $s = 1 - \sum A$, so the conditions of Definition 20 hold; the largest number of widths in a row is 8. The coefficient vectors c multiply $P_2, P_4, \dots$ in order. The first six families are as follows.

(R0) One deterministic row, $A = \varnothing$, $s = 1$, with $c \in \mathbb{Z}^{19}$ given by

$$(1132213,\ 308439,\ -659459,\ 149443,\ -294892,\ -26906,\ -225320)$$

for $P_2, \dots, P_{14}$, followed by

$$(-29178,\ -49586,\ 38499,\ 143,\ 39925,\ -9921,\\ 27126,\ -14745,\ 19252,\ -16741,\ 12735,\ -14251)$$

for $P_{16}, \dots, P_{38}$.

(R1) Linear rows $A = (l/640)$, $1 \le l \le 639$, with $c = (c_{l,1}, \dots, c_{l,7}) \in \mathbb{Z}^7$ defined below; and, for the nine values $i \in \{5, 20, 40, 60, 80, 100, 120, 140, 155\}$, linear rows $A = (i/160)$ whose coefficient vectors have zeros for $P_2, \dots, P_{14}$ followed by four integers for $P_{16}, \dots, P_{22}$, listed in Table 1.

(R2) Forty-seven quadratic rows $A = (a, b)$ with $a, b \in \frac{1}{40}\mathbb{Z}$, $a + b \le \frac{34}{40}$, and $c \in \mathbb{Z}^5$, listed in Table 7.

(R3) Fourteen cubic rows with $c \in \mathbb{Q}^3$, listed in Table 2.

(R4) Six quartic rows $A = (a, a, a, a)$ with $c \in \mathbb{Z}^2$, and

(R5) six quintic rows with $c \in \mathbb{Z}^2$, listed in Table 3.

The linear coefficients. Let $(w_{i,j})_{1 \le i \le 159,\, 1 \le j \le 7}$ be the integer array of linear coefficients published with the programs at `https://github.com/charlieoneill11/maths-releases`, and put $w_{0,j} = w_{160,j} = 0$. Let $a_{i,j} = w_{i,j}$ if $8 \nmid i$ and $a_{i,j} = \frac{1}{2}(w_{i-1,j} + w_{i+1,j})$ if $8 \mid i$. Start from $c_{l,j} = 0$ for $0 \le l \le 640$. For each $1 \le i \le 159$ and $1 \le j \le 7$, add $w_{i,j}$ to $c_{4i,j}$, and for $k = 1, 2, 3$ let β be the integer nearest to $a_{i,j}(4 - k)/16$ (ties to even), add β to $c_{4i-k,j}$ and to $c_{4i+k,j}$, and subtract 2β from $c_{4i,j}$. This spreads each coefficient

Table 1: Higher Legendre modes of the linear rows in (R1): coefficients of $P_{16}, P_{18}, P_{20}, P_{22}$ for $A = (i/160)$.

i					i				
5	-870168	89725	33939	97095	100	16892	-1307	8248	5824
20	-270416	176980	-33503	39115	120	78	1409	-2550	1369
40	-21424	55884	8610	50448	140	1127	1531	5319	3412
60	45099	2172	32428	41136	155	-2324	148	5272	4760
80	39661	-4705	24324	18822					

of a $\frac{1}{160}$-grid over its neighbours on the $\frac{1}{640}$-grid while preserving the total in each Legendre degree.

The remaining four families are as follows. For $j \geq 1$ write e_j for the coefficient vector of P_{2j} alone, so that $\ell_{s,e_j}(u) = P_{2j}(u/s)\mathbf{1}_{[-s,s]}(u)$.

(R6) Nine rows $A = (a, \dots, a)$ (k equal widths, $k \in \{6, 7\}$) with profile $w\,\ell_{s,e_j}$, listed in Table 4.

(R7) Thirteen rows with four or eight widths and profile $w\,\ell_{s,e_j}$, listed in Table 5.

(R8) One hundred and three rows with $r \in \{0, 1, 2\}$ and profile $10^{-9}\ell_{s,c}$, $c \in \mathbb{Z}^6$ (if $r \leq 1$) or $c \in \mathbb{Z}^{\leq 3}$ (if $r = 2$), listed in Table 8 in Appendix A. There is one row with $r = 0$, nineteen with $r = 1$ and eighty-three with $r = 2$.

(R9) Twenty-six rows with $r = 3$ and profile $10^{-9}\ell_{s,c}$, $c \in \mathbb{Z}^2$, listed in Table 6.

In all four families the widths are multiples of $\frac{1}{160}$ with $\sum A \leq \frac{152}{160}$.

All coefficients were found by numerical optimisation, which plays no role in the proof.

8.2 The certificate

Let F_1 consist of the rows (R0)–(R2), and let F_2, F_3, F_4 add the rows (R3), (R4), (R5) in turn. Put

$$z = 1014885461 \cdot 10^{-16},$$

Table 2: Cubic rows (R3): widths A (in units of $\frac{1}{160}$) and coefficients of P_2, P_4, P_6.

$160A$	c_1	c_2	c_3
$(16,16,16)$	$-\frac{843632985}{1024}$	$\frac{4527783555}{2048}$	$-\frac{1382901975}{1024}$
$(24,24,24)$	$-\frac{2156795735}{4096}$	$\frac{3256426715}{2048}$	$-\frac{2836411325}{4096}$
$(28,28,28)$	$\frac{150750809}{256}$	$\frac{185921351}{256}$	$-\frac{54735791}{128}$
$(32,32,32)$	$\frac{1709829173}{4096}$	$\frac{2969684575}{4096}$	$-\frac{76174163}{512}$
$(36,36,36)$	$\frac{282792425}{512}$	$\frac{3214325}{16}$	$\frac{75487905}{512}$
$(40,40,40)$	$\frac{90726253}{256}$	$\frac{54642395}{1024}$	$-\frac{38032237}{512}$
$(44,44,44)$	$\frac{694679571}{4096}$	$-\frac{8283357}{512}$	$-\frac{88300827}{2048}$
$(48,48,48)$	$\frac{10795455}{256}$	$-\frac{230175}{512}$	$\frac{4070385}{512}$
$(16,20,84)$	$-\frac{260794455}{512}$	$\frac{27361435}{512}$	$-\frac{921335}{16}$
$(16,32,80)$	$-\frac{2148009149}{4096}$	$\frac{85191821}{2048}$	$\frac{230378291}{4096}$
$(8,20,92)$	$-\frac{1150920225}{1024}$	$\frac{342556025}{512}$	$-\frac{222565425}{1024}$
$(12,32,88)$	$-\frac{300094123}{4096}$	$\frac{500910361}{4096}$	$-\frac{133210399}{2048}$
$(24,40,64)$	$\frac{287061755}{2048}$	$\frac{334905713}{2048}$	$-\frac{5247569}{512}$
$(24,32,56)$	$\frac{4299524187}{4096}$	$\frac{947160299}{4096}$	$-\frac{462839355}{2048}$

let F_5 consist of the rows of F_4 with their profiles multiplied by z, together with the rows (R6), and let $F_6, F_7, F_8 = F$ add the rows (R7), (R8), (R9) in turn. Put $p_0 = (19 - 3.916317365)/16$ as in Lemma 31,

$$\begin{aligned} p_1 &= 0.94531427448210, & p_2 &= 0.945321516147, \\ p_3 &= 0.945325433280444, & p_4 &= 0.945325784098260, \\ p_5 &= 0.94532732177059, & p_6 &= 0.9453286989, \\ p_7 &= 0.9453391726, & p_8 &= p_* = 0.9453419890, \end{aligned}$$

and, for an amplitude F',

$$\mathcal{Q}(F') = p_0 + \langle G, F' \rangle - \tfrac{1}{2}\langle F', F' \rangle.$$

Proposition 32. *1. For $\nu = 1,2,3,4$ we have $\langle G, F_\nu \rangle > 0$, $\langle F_\nu, F_\nu \rangle > 0$ and*

$$p_0 + \frac{\langle G, F_\nu \rangle^2}{2\langle F_\nu, F_\nu \rangle} > p_\nu.$$

2. For $\nu = 5,6,7,8$ we have $\mathcal{Q}(F_\nu) > p_\nu$.

Table 3: Quartic rows (R4) and quintic rows (R5): widths in units of $\frac{1}{160}$, coefficients of P_2, P_4.

$160A$	c_1	c_2	$160A$	c_1	c_2
$(16)^4$	-510386	-291874	$(16)^5$	1779892	-8867799
$(20)^4$	-1974752	1240532	$(20)^5$	4580	-5962351
$(24)^4$	1295990	-2402543	$(24)^5$	-4232853	814887
$(28)^4$	2797966	-473911	$(28)^5$	-1185699	-85106
$(32)^4$	1264735	124269	$(8, 12, 16, 24, 64)$	22938237	0
$(36)^4$	211240	-41256	$(12, 16, 20, 32, 48)$	-11687453	0

Table 4: Rows (R6): k equal widths a, the Legendre degree $2j$, and the weight w.

k	$160a$	$2j$	$10^7 w$
6	20	2	-53945511
6	20	4	45352129
7	16	2	-11243176
6	18	2	9261068
6	20	6	-7416954
6	16	4	36415509
6	24	2	-8230813
6	24	4	939245
7	20	2	11527588

Proof. Every quantity below is a finite sum of restricted cumulants given by Lemmas 21 and 25 and Corollary 26, and each cumulant is evaluated exactly in rational arithmetic by Lemmas 27 and 28, except that H is replaced by H^* in the products $G\check{\eta}_k$. The programs listed in Appendix A carry out these computations; they use only integer and rational arithmetic, and every comparison is an exact comparison of rationals.

The central products. By Lemma 21 the row (R0) contributes 0 to $\langle G, F\rangle$, and by Corollary 26 each other row contributes the cumulant $\mathcal{K}(G\check{\eta}_k; T_{A_k})$. The program computes $\mathcal{K}(G^*\check{\eta}_k; T_{A_k})$ exactly, and Lemma 30 gives an interval of rational numbers containing $\langle G, k\rangle$. Summing gives rational enclosures $C_\nu^- < \langle G, F_\nu\rangle < C_\nu^+$ with $C_\nu^- > 0$.

Stage $\nu = 1$. The program evaluates $\langle k, l\rangle$ exactly for all pairs of rows of F_1 (no approximation is involved, as G does not occur), groups them by the first row, rounds each group upwards to a multiple of 10^{-8}, and

Table 5: Rows (R7): widths in units of $\frac{1}{160}$ (an exponent denotes repetition), the Legendre degree $2j$, and the weight w.

$160A$	$2j$	$10^8 w$
(8,16,32,64)	2	30312549
(8,16,32,64)	4	-5245954
(8,20,32,68)	2	-62542918
(12,16,24,64)	2	-66526833
(12,16,24,64)	4	59046867
(12,20,32,60)	2	-27804655
(12,20,32,60)	4	63571537
$(14)^8$	4	-5630338346
$(14)^8$	6	7851010526
$(16)^8$	2	5266380051
$(16)^8$	4	-4222736926
(16,24,40,48)	2	49847039
$(18)^8$	2	1657319238

adds the results to obtain $D_1^+ \geq \langle F_1, F_1\rangle$. It then checks $D_1^+ > 0$ and $p_0 + (C_1^-)^2/(2D_1^+) > p_1$.

Stages $\nu = 2, 3, 4$. Let $g_{\nu-1} = p_{\nu-1} - p_0$. By the previous stage, $\langle G, F_{\nu-1}\rangle^2 \geq 2g_{\nu-1}\langle F_{\nu-1}, F_{\nu-1}\rangle$. By Lemma 23, with F'' the new rows,

$$\langle F_\nu, F_\nu\rangle \leq D_\nu^+ := \frac{(C_{\nu-1}^+)^2}{2g_{\nu-1}} + 2\langle F_{\nu-1}, F''\rangle + \langle F'', F''\rangle,$$

where the last two terms are computed exactly. The program checks $D_\nu^+ > 0$ and $p_0 + (C_\nu^-)^2/(2D_\nu^+) > p_\nu$. Since $\langle F_\nu, F_\nu\rangle \geq 0$ (Remark after Theorem 22) and $\langle G, F_\nu\rangle > 0$, Cauchy–Schwarz in the form $\langle G, F_\nu\rangle^2 \leq \lambda\langle F_\nu, F_\nu\rangle$ gives $\langle F_\nu, F_\nu\rangle > 0$. This proves (1).

Stage $\nu = 5$. Write the rows (R6) as $H_1, \ldots, H_9$, regarded as one-row amplitudes. By bilinearity,

$$\mathcal{Q}(F_5) = p_0 + z\langle G, F_4\rangle + \sum_i \langle G, H_i\rangle - \frac{1}{2}\Big(z^2\langle F_4, F_4\rangle + 2z\sum_i \langle F_4, H_i\rangle + \sum_{i,j}\langle H_i, H_j\rangle\Big).$$

Since $z > 0$, we have $z\langle G, F_4\rangle > zC_4^-$. Stage 4 gives $\langle F_4, F_4\rangle < \langle G, F_4\rangle^2/(2g_4) < (C_4^+)^2/(2g_4)$ with $g_4 = p_4 - p_0$. For each i, Lemma 30 gives a rational interval containing $\langle G, H_i\rangle$, and we use its lower endpoint. (The

Table 6: Rows (R9): widths in units of $\frac{1}{160}$ and the coefficients c_1, c_2 of P_2, P_4; the profile is $10^{-9}\ell_{s,c}$.

$160A$	c_1	c_2	$160A$	c_1	c_2
(8,12,24)	−121717517	491610893	(16,32,80)	20845072	−13504451
(8,16,80)	−14064566	−132741707	(16,44,64)	28046350	21683205
(8,24,64)	−26652553	174310795	(20,20,60)	12736579	66406823
(8,28,48)	−66504478	127384262	(20,36,52)	93880390	−6395813
(8,48,64)	38776636	−2023335	(20,36,68)	−11293709	−1725090
(12,24,48)	82266724	130618646	(24,24,64)	−5068851	−6379821
(12,28,60)	185849866	11551660	(24,32,72)	−43669224	115682
(12,32,40)	−116997456	284382384	(24,40,56)	−5891849	−3126459
(12,36,84)	−3460378	2595712	(24,48,64)	6578063	2318498
(12,40,64)	70680011	47058362	(32,32,64)	29119324	−19344054
(12,52,72)	19681327	−1896137	(32,40,48)	27089147	−1290595
(16,24,72)	−33517020	−53977051	(32,40,64)	−18632458	2244473
(16,32,56)	−15400926	156214903	(32,48,64)	−2867142	907756

program encloses $\langle G, \cdot\rangle$ for the row with profile ℓ_{s,e_j} and multiplies by w, taking the lower or the upper endpoint according as $w > 0$ or $w < 0$.) The Gram entries $\langle F_4, H_i\rangle = \sum_k \langle k, H_i\rangle$, summed over the rows k of F_4, and $\langle H_i, H_j\rangle$ are evaluated exactly by Lemma 25. The result is a rational number $R_5 < \mathcal{Q}(F_5)$, and the program checks $R_5 > p_5$.

Stages $\nu = 6, 7, 8$. For amplitudes F' and H,

$$\mathcal{Q}(F' + H) = \mathcal{Q}(F') + \langle G, H\rangle - \langle F', H\rangle - \tfrac{1}{2}\langle H, H\rangle. \tag{13}$$

The program starts from the bound $\mathcal{Q}(F_5) > p_5$ of stage 5 and adds the rows of (R7), (R8), (R9) one at a time, in the order of Table 5, then Table 8, then Table 6. At each step F' is the current amplitude and H the new row. A lower bound for $\langle G, H\rangle$ is 0 if H has no widths (Lemma 21), and otherwise the endpoint of the interval of Lemma 30 chosen as in stage 5. The term $\langle F', H\rangle$ is z times the sum of $\langle k, H\rangle$ over the rows k of F_4, plus the sum of $\langle H', H\rangle$ over the rows H' of (R6)–(R9) added before H; it and $\langle H, H\rangle$ are evaluated exactly by Lemma 25. By (13) the running total remains a lower bound for $\mathcal{Q}$ of the current amplitude. After the last row of (R7), (R8) and (R9) the running totals R_6, R_7, R_8 satisfy $R_\nu > p_\nu$. This proves (2). □

The enclosures produced by the programs are, in decimal approximation,

ν	C_ν^-	C_ν^+	D_ν^+	$p_0 + (C_\nu^-)^2/(2D_\nu^+)$
1	50553.4662326	50553.4664934	$4.94493878215 \times 10^{11}$	$0.945314274482106\ldots$
2	50636.8088192	50636.8090970	$4.94739221446 \times 10^{11}$	$0.945321516147273\ldots$
3	51074.7317985	51074.7320816	$5.02573844454 \times 10^{11}$	$0.945325433280444\ldots$
4	51079.4486250	51079.4489122	$5.02598736360 \times 10^{11}$	$0.945325784098260\ldots$

The upper enclosure C_4^+ is computed in stage 5. For the later stages the running totals are

ν	R_ν	$R_\nu - p_\nu$
5	$0.945327321770591656\ldots$	$1.65 \cdot 10^{-15}$
6	$0.945328698910093200\ldots$	$1.00 \cdot 10^{-11}$
7	$0.945339172651395717\ldots$	$5.13 \cdot 10^{-11}$
8	$0.945341989313383391\ldots$	$3.13 \cdot 10^{-10}$

Several margins are close to the precision of the decimals shown, so these decimals cannot replace the exact comparisons. For example, in stage 3 the program shows that the margin lies strictly between two consecutive multiples of 10^{-40}, and in stage 5 the margin is $1.656\ldots \cdot 10^{-15}$.

Proof of Theorem 1. Let $\mathcal{D}$ be one of the six classes and $p' = \mathcal{Q}(F) = p_0 + \langle G, F\rangle - \frac{1}{2}\langle F, F\rangle$, so $p' > p_*$ by Proposition 32(2) with $\nu = 8$. By Theorem 22 with the empty pattern and Lemma 31, the lower limit of the W-weighted nonvanishing proportion in $\mathcal{D}$ is at least $(19 - \cot\frac{1}{4})/16 + \langle G, F\rangle - \frac{1}{2}\langle F, F\rangle > p'$, for every admissible W.

Let $\varepsilon > 0$ and choose W with $0 \le W \le \mathbf{1}_{(1,2)}$ and $\int(\mathbf{1}_{(1,2)} - W) < \varepsilon$. For $Y \ge 1$, the number of $d \in \mathcal{D}$ with $Y < m(d) \le 2Y$ and $L(\frac{1}{2}, \chi_d) = 0$ is at most

$$\sum_{d\in\mathcal{D}} W\Big(\frac{m(d)}{Y}\Big)\mathbf{1}[L(\tfrac{1}{2},\chi_d) = 0] + \sum_{d\in\mathcal{D}} \Big(\mathbf{1}_{(1,2]} - W\Big)\Big(\frac{m(d)}{Y}\Big) \le \big(1 - p' + o(1)\big)\mathfrak{M}(Y) + (c_{\mathcal{D}}\varepsilon + o(1))Y,$$

where $\mathfrak{M}(Y)$ is the mass of Lemma 8 (empty pattern) and $c_{\mathcal{D}}Y$ is the asymptotic number of $d \in \mathcal{D}$ with $Y < m(d) \le 2Y$, by the elementary count of squarefree integers in progressions. As $\mathfrak{M}(Y) \le (1 + o(1))c_{\mathcal{D}}Y$, the proportion of vanishing in the sharp shell is at most $1 - p' + O(\varepsilon) + o(1)$. Now decompose $\{d \in \mathcal{D} : |d| \le X\}$ into the shells $X/2^{j+1} < |d| \le X/2^j$, $0 \le j < J$, and a remainder of relative size $O(2^{-J})$. Letting $X \to \infty$, then $\varepsilon \to 0$ and $J \to \infty$, the lower limit of the nonvanishing proportion in $\mathcal{D}$ is at least $p' > p_*$. The six classes partition the fundamental discriminants $d \ne 1$,

and the proportion of each class among $1 < |d| \le X$ converges; so the same bound holds for all d. □

9 Prescribed local conditions: proof of Theorem 2

Proof of Theorem 2. Let $\mathcal{D}$ be a class and $z = o(\log X)$. Since $|d| = \kappa m(d)$, the shell $X < |d| \le 2X$ is the shell $X/\kappa < m \le 2X/\kappa$, and $z = o(\log(X/\kappa))$. Lemma 8 shows that every pattern occurs for large X, uniformly. Theorem 22 applies with the amplitude of Section 8 and is uniform in the pattern, so the W-weighted nonvanishing proportion in $\mathcal{D}_v$ is at least $p' - o(1)$ uniformly in v, with $p' > p_*$ as above. The passage from W to the sharp shell is as in the proof of Theorem 1. Let $\varepsilon > 0$ and choose W with $0 \le W \le \mathbf{1}_{(1,2)}$ and $\int(\mathbf{1}_{(1,2)} - W) < \varepsilon$. The weights $\mathbf{1}_{(1,2]}$ and $\mathbf{1}_{(1,2]} - W$ are fixed, of bounded variation and supported in $[1,2]$, so by Lemma 8 and the remark at the end of its proof (with X/κ in place of X) their masses on $\mathcal{D}_v$ are $(1+o(1))\,\theta_\kappa \mathcal{A}(X/\kappa)\int(\cdot)/(r2^t)$, uniformly in v. The first is the number of members of $\mathcal{D}_v$ in the shell. Hence the number of those with $L(\frac{1}{2}, \chi_d) = 0$ is at most

$$\big(1 - p' + o(1)\big)\mathfrak{M}_v(X/\kappa) + \mathfrak{M}_v^{\mathbf{1}_{(1,2]} - W}(X/\kappa) \\ \le \big(1 - p' + \varepsilon + o(1)\big)\,\#\{d \in \mathcal{D}_v : X < |d| \le 2X\},$$

uniformly in v, where $\mathfrak{M}_v^{\mathbf{1}_{(1,2]} - W}$ denotes the mass with that weight. Taking $\varepsilon < p' - p_*$ proves the first statement.

For the second, let $b < p_*$ and suppose no c_b exists. Then for each $j \ge 1$ there are arbitrarily large X (we choose X_j increasing to infinity with $(\log X_j)/j \ge 3$) and a class and a pattern on the odd primes $p \le (\log X_j)/j$ which either does not occur in the shell or has nonvanishing proportion below b. Since there are six classes, one class occurs for infinitely many j; pass to that subsequence. Define $z(X) = (\log X_j)/j$ if $X = X_j$ and $z(X) = 3$ otherwise. Then $z = o(\log X)$, and along X_j the failing patterns contradict the uniform statement just proved. □

Averaging over atoms, the bound in Theorem 2 holds for every nonempty union of patterns, even one that varies with X.

10 The two-moment ceiling: proof of Theorem 3

In this section m runs over odd squarefree positive integers, weighted by $\Phi(m/X)$ and normalised to a probability measure; $\mathbb{E}$ and $\Pr$ refer to it. Write

$\chi = \chi_{8m}$, $Z = Z_m = L(\frac12, \chi_{8m})$, and h for odd positive integers. Everything in this section except the final remark uses neither the Riemann hypothesis nor GRH.

10.1 Local factors and twisted moments

For an odd prime p let x be a random variable with $\Pr(x=0) = r_p(0) = 1/(p+1)$ and $\Pr(x=\pm1) = r_p(\pm1) = p/(2p+2)$; this is the limiting law of $\chi(p)$. Put $x^0 = 1$. For $j = 0, 1, 2$ let

$$U_{j,p} = \mathbb{E}\Big[\Big(1 - \frac{x}{\sqrt{p}}\Big)^{-j}\Big], \qquad a_j(h) = \prod_{p^e \| h} \frac{\mathbb{E}[x^e(1 - x/\sqrt{p})^{-j}]}{U_{j,p}},$$
$$A_{j,v} = \prod_{2<p\le y} \frac{r_p(v_p)(1 - v_p/\sqrt{p})^{-j}}{U_{j,p}}$$

for a pattern v on $2 < p \le y$. Thus $a_0(h) = \mathbf{1}_{h=\square} \prod_{p\mid h} \frac{p}{p+1}$, and for $e \ge 1$, $a_1(p^e) = \beta_p p^{-(e \bmod 2)/2}$ with $\beta_p = p^2/(p^2+p-1)$. A direct computation gives $(1-\frac1p)U_{1,p} = 1 - \frac{1}{p(p+1)}$ and $(1-\frac1p)^2 U_{2,p} = h(p) := 1 + \frac1p + \frac{1}{p^2} - \frac{4}{p(p+1)}$; in particular $U_{1,p} = 1 + \frac1p + O(p^{-2})$ and $U_{2,p} = 1 + \frac3p + O(p^{-2})$.

Lemma 33 (Soundararajan)**.** *Let $\varepsilon > 0$. For odd h, write h_1 for the product of the primes dividing h to an odd power. Then*

$$\mathbb{E}[\chi(h)] = a_0(h) + O_\varepsilon(X^{-1/2+\varepsilon} h^{1/2}),$$
$$\mathbb{E}[Z\chi(h)] = K_1 a_1(h)\Big(L - 2\log h_1 + C_0 + \sum_{p\mid h} C(p)\frac{\log p}{p}\Big) + O_\varepsilon(X^{-1/4+\varepsilon} h^{1/2+\varepsilon}),$$
$$\mathbb{E}[Z^2\chi(h)] = K_2 a_2(h)\, \mathcal{P}_2(L; h) + O_\varepsilon(X^{-1/4+\varepsilon} h^{1/2+\varepsilon}),$$

with $K_1 = \frac14 \prod_{p>2}(1-\frac1p)U_{1,p}$, $K_2 = \frac{1}{192}\prod_{p>2}(1-\frac1p)^3 U_{2,p} = \frac{1}{192}\prod_{p>2}(1-\frac1p)h(p)$, constants C_0 and $C(p) = O(1)$ depending only on Φ, and

$$\mathcal{P}_2(L; h) = (L - \log h_1)^3 - 3(L - \log h_1)\sum_{p\mid h_1}\log^2 p + C_1 \sum_{p\mid h_1}\log^2 p + \sum_{i,k\le 3}\sum_{\substack{\varsigma\mid h\\ \varrho\mid h_1}} \frac{\Lambda_i(\varsigma)\Lambda_k(\varrho)}{\varsigma\varrho} D(\varsigma,\varrho) Q_{ik}(L - \log h_1),$$

where Q_{ik} are polynomials of degree ≤ 2 depending only on Φ, $D(\varsigma,\varrho)$ is bounded, and Λ_i is supported on integers with at most i distinct prime factors and satisfies $0 \le \Lambda_i(n) \ll (1+\log n)^i$, $\Lambda_0 = \mathbf{1}_{\{1\}}$, $\sum_{n\mid h}\Lambda_i(n) = (\log h)^i$.

Proof. The first line follows from squarefree orthogonality and the Pólya–Vinogradov inequality; it is [5, Lemma 3.2]. The second line is the twisted first moment [16, Proposition 1.2], in the form [5, Lemma 3.3]. The third line is the twisted second moment [16, Proposition 1.3]. In both we take the square-divisor cutoff $Y = X^{1/4}$ and bound the contribution of the non-squarefree terms by [16, Proposition 1.1] with the trivial mollifier, since $|\chi(h)| \le 1$. We have divided by the family mass $\frac{4}{\pi^2}X\int\Phi\,(1+O(X^{-1/2}))$ and rewritten the arithmetic factors in terms of $a_j(h)$, which is a local computation at each prime. In the arXiv version of [16], the product defining the leading constant D in Proposition 1.3 is printed over $p>3$; we use the product over all $p \ge 3$, as in K_2. The leading term comes from the diagonal terms $n = k^2$, k odd, of the approximate functional equation for $L(\frac12,\chi_{8m})^2$. Their Dirichlet series, divided by $(1-2^{-1-2s})^3\zeta(1+2s)^3$, has at $s=0$ the Euler factor $(1-\frac1p)^3U_{2,p} = (1-\frac1p)h(p)$ at every odd prime p, and the prime 3 plays no special role. The same computation for the first moment gives the constant C of [16, Proposition 1.2], whose product is printed over $p \ge 3$. □

10.2 Patterns

Let $P = \prod_{2<p\le y} p$, so $\log P \le (1+o(1))y$. The indicator of the pattern v is $\mathbf{1}_v = \prod_{2<p\le y}\iota_p$ with $\iota_p = 1-\chi(p)^2$ if $v_p = 0$ and $\iota_p = \frac12(\chi(p)^2 + v_p\chi(p))$ if $v_p = \pm1$. Expanding, $\mathbf{1}_v = \sum_{h\mid P^2} b_v(h)\chi(h)$ with $\sum_h |b_v(h)| \le 2^{\pi(y)}$. Since ι_p evaluated at x is $\mathbf{1}_{x=v_p}$, multiplicativity gives

$$\sum_h b_v(h)a_j(h) = A_{j,v} \qquad (j=0,1,2). \tag{14}$$

Also $A_{j,v} \ge \mathrm{e}^{-O(y)}$. By Lemma 33, $\Pr(v) = A_{0,v} + O(2^{\pi(y)}PX^{-1/2+\varepsilon}) = A_{0,v}(1+O(X^{-1/3}))$ when $y \le cL$ with c small. In particular every pattern has positive probability. We fix c small enough for all the requirements below; they are finitely many and independent of X.

Local weights. In a sum $\sum_h b_v(h)a_j(h)\Omega(h)$, where Ω marks a condition on the exponent $e = v_p(h)$ at one prime p, the marked sum equals $A_{j,v}$ times a local factor, the ratio of the marked to the unmarked local sum at p. Direct computation gives: for $e \ge 1$ or $e = 2$, the factor is $O(1)$ if $v_p = \pm1$ and $O(p)$ if $v_p = 0$; for $e = 1$, it is $O(p^{-1/2})$ if $v_p = \pm1$ and 0 if $v_p = 0$. Conditions at distinct primes multiply.

10.3 The first moment on every pattern

Let $T(v) = \sum_{2<p\le y,\, v_p=0} \log p$, and let $n \le N = X^{1/100}$ be squarefree with all prime factors greater than y.

Lemma 34. *Uniformly in such n, in $3 \le y \le cL$ and in v,*

$$\frac{\mathbb{E}[Z\chi(n)\mathbf{1}_v]}{K_1 A_{1,v} a_1(n)} = L - 2\log n + O_\Phi\Big(1 + \sqrt{y} + T(v) + \sum_{p\mid n} \frac{\log p}{p}\Big) + O(X^{-1/10}).$$

In particular the left side lies between $L/2$ and $2L$.

Proof. Write $\mathbb{E}[Z\chi(n)\mathbf{1}_v] = \sum_h b_v(h)\mathbb{E}[Z\chi(hn)]$ and apply Lemma 33. Since $(h,n)=1$ we have $a_1(hn) = a_1(h)a_1(n)$ and $(hn)_1 = h_1 n$. By (14), the terms of the polynomial not depending on h give $A_{1,v}a_1(n)(L - 2\log n + C_0 + \sum_{p\mid n} C(p)\frac{\log p}{p})$. The term $-2\log h_1 = -2\sum_{p\le y}\log p\,\mathbf{1}_{p\mid h_1}$ costs, by the local weights, $O(A_{1,v}\sum_{p\le y}\log p/\sqrt{p}) = O(A_{1,v}\sqrt{y})$, and $\sum_{p\mid h} C(p)\frac{\log p}{p}$ costs $O(A_{1,v}(\log y + T(v)))$. The power errors total $O(2^{\pi(y)}P^{1+2\varepsilon}N^{1/2+\varepsilon}X^{-1/4+\varepsilon})$, which is $O(X^{-1/10}A_{1,v}a_1(n))$ because $A_{1,v} \ge X^{-O(c)}$ and $a_1(n) \gg n^{-1/2-\varepsilon}$. For the last assertion: $2\log n \le L/50$; $\sqrt{y} + T(v) \ll \sqrt{cL} + cL$; and $\sum_{p\mid n}\frac{\log p}{p} \le \sum_{p\le\sqrt{L}}\frac{\log p}{p} + \omega(n)\frac{\log\sqrt{L}}{\sqrt{L}} \ll \log L + \sqrt{L}$. □

10.4 Proof of part (a)

Let $\mathcal{N}$ be the set of squarefree $n \le N$ with all prime factors greater than y, $B = \sum_{n\in\mathcal{N}}\chi(n)n^{-1/2}$ and $S = \sum_{n\in\mathcal{N}} n^{-1}$. For distinct $n, n' \in \mathcal{N}$ and $h \mid P^2$, hnn' is not a square, so by Lemma 33 the off-diagonal part of $\mathbb{E}[B^2\mathbf{1}_v]$ is $O(2^{\pi(y)}PN^2X^{-1/2+\varepsilon}) = o(\Pr(v))$; the diagonal is at most $S\Pr(v)$. Hence $\mathbb{E}[B^2 \mid v] \le S + o(1)$. By Lemma 34, for $n \in \mathcal{N}$, $\mathbb{E}[Z\chi(n)\mathbf{1}_v] \ge \frac14\mathbb{E}[Z\mathbf{1}_v]a_1(n)$, and $a_1(n)n^{-1/2} = \prod_{p\mid n}\beta_p/p$. So

$$\mathbb{E}[ZB \mid v] \ge \frac14\mathbb{E}[Z \mid v]\sum_{n\in\mathcal{N}}\frac{1}{n}\prod_{p\mid n}\beta_p \gg \mathbb{E}[Z \mid v]\,S.$$

For the last step note $\log\beta_p \ge -2/p$, and under the probability measure on $\mathcal{N}$ proportional to $1/n$ the mean of $\sum_{p\mid n}1/p$ is at most $\sum_p p^{-2}$, since $\sum_{n\in\mathcal{N},\,p\mid n}\frac{1}{n} \le \frac1p\sum_{n\in\mathcal{N}}\frac1n$; apply Jensen's inequality to $\exp$.

We have $S \asymp \log N/\log(2y)$. The upper bound is $S \le \prod_{y<p\le N}(1+\frac1p)$ and Mertens' theorem. For the lower bound fix a small $\delta > 0$ and consider the measure on squarefree n with prime factors in $(y, N^\delta]$ proportional to

$1/n$; its total mass is $\prod_{y<p\le N^\delta}(1+\frac{1}{p}) \gg \delta\log N/\log(2y)$ and the mean of $\log n$ is $\sum_{y<p\le N^\delta}\frac{\log p}{p+1}\le 2\delta\log N$, so by Markov's inequality at least half of the mass lies on $n\le N$ when $\delta\le\frac14$.

By Cauchy–Schwarz, $\mathbb{E}[Z\mid v]^2S^2 \ll \mathbb{E}[ZB\mid v]^2 \le \mathbb{E}[Z^2\mid v](S+o(1))$, so $\mathbb{E}[Z\mid v]^2 \ll \mathbb{E}[Z^2\mid v]/S \ll \mathbb{E}[Z^2\mid v]\log(2y)/L$. □

10.5 The second moment on most patterns

Lemma 35. *Fix $c<\frac14$ small. Uniformly for $3\le y\le cL$ and patterns with $T(v)\le(\log L)^3$,*

$$\mathbb{E}[Z^2\mathbf{1}_v] = K_2L^3A_{2,v}(1+o(1)), \qquad \mathbb{E}[Z\mathbf{1}_v] = K_1LA_{1,v}(1+o(1)).$$

Proof. The second statement is Lemma 34 with $n=1$, since $\sqrt{y}+T(v)=o(L)$. For the first, write $\mathbb{E}[Z^2\mathbf{1}_v]=\sum_{b\mid P^2}b_v(b)\mathbb{E}[Z^2\chi(b)]$ and use Lemma 33. The power errors total $X^{-1/4+c+o(1)+O(\varepsilon)}$, which is $o(L^3A_{2,v})$ because $A_{2,v}\ge\exp(-O(\pi(y)+T(v)))=X^{-o(1)}$ on these patterns. The leading term $(L-\log b_1)^3$ expands as L^3 plus terms $L^{3-r}(\log b_1)^r$, $1\le r\le 3$. By (14) the term L^3 gives $K_2L^3A_{2,v}$. We show that every other term is $o(L^3A_{2,v})$ after summation against $b_v(b)a_2(b)$. Taking absolute values before summing over b would lose this, so we use the local weights.

Logarithmic marks. $(\log b_1)^r=\sum_{q_1,\dots,q_r\le y}\prod_i\log q_i\,\mathbf{1}_{q_i\parallel_{\mathrm{odd}}b}$. Group the indices by equal primes; a block of size d at a prime q carries the condition $e=1$ (as $b\mid P^2$), with local factor $O(q^{-1/2})$ or 0, and weight $(\log q)^d$. Summing, the block contributes $O(\sum_{q\le y}(\log q)^d/\sqrt{q})=O(\sqrt{y}(1+\log y)^d)$. Hence $\sum_b b_v(b)a_2(b)(\log b_1)^r=O(A_{2,v}(1+\sqrt{y})^r(1+\log y)^r)$, and the terms $L^{3-r}(\log b_1)^r$ contribute $o(L^3A_{2,v})$. The term $\sum_{p\mid b_1}\log^2 p$ is treated in the same way, giving $O(A_{2,v}(L\sqrt{y}+y)(1+\log y)^3)$.

Divisor marks. Fix $\varsigma\mid b$ and $\varrho\mid b_1$; the set J of their prime factors has at most six elements. For $p\mid\varsigma$ the condition is $e\ge v_p(\varsigma)$, and for $p\mid\varrho$ it is $e=1$. After including the factor $1/(\varsigma\varrho)$, the cost at $p\in J$ is $O(1/p)$ if $v_p=\pm1$; if $v_p=0$ the only possibility is $p\mid\varsigma$, $p\nmid\varrho$, with cost $O(1)$ since $p^{v_p(\varsigma)}\ge p$. As $\log(\varsigma\varrho)\le 18\log y$ on the support, the divisor marks, including the factors $\Lambda_i\Lambda_k$, cost at most

$$C(1+\log y)^6\Big(1+\sum_{p\le y}\frac{1}{p}+\#\{p:v_p=0\}\Big)^6 \ll (1+\log L)^{C'}$$

relative to $A_{2,v}$, because $\#\{p:v_p=0\}\le T(v)/\log 3$. Combined with logarithmic marks from $Q_{ik}(L-\log b_1)$ as above, these terms contribute $O(A_{2,v}L^2(1+\log L)^{C''})=o(L^3A_{2,v})$. □

Lemma 36. *For every fixed $A > 0$, $\Pr(T(v) > (\log L)^3) = O_A(L^{-A})$ uniformly for $3 \le y \le L$, where T is evaluated at the pattern of m.*

Proof. For squarefree $\delta \mid P$, $\Pr(\delta \mid m) \ll 1/\delta$ by counting multiples of δ up to $2X$. Hence $\mathbb{E}[\mathrm{e}^{T/\log y}] \le \sum_{\delta \mid P} \prod_{p \mid \delta} (p^{1/\log y} - 1) \Pr(\delta \mid m) \le \prod_{p \le y} \left(1 + \frac{C(e-1)\log p}{p \log y}\right) \ll 1$. By Markov's inequality, $\Pr(T > (\log L)^3) \ll \exp(-(\log L)^3/\log y) \le \exp(-(\log L)^2)$. □

10.6 Proof of parts (b) and (c)

Call a pattern *good* if $T(v) \le (\log L)^3$. For a good pattern, Lemma 35 and $\Pr(v) = A_{0,v}(1 + o(1))$ give

$$\frac{\mathbb{E}[Z \mid v]^2}{\mathbb{E}[Z^2 \mid v]} = \frac{\mathbb{E}[Z\mathbf{1}_v]^2}{\mathbb{E}[Z^2\mathbf{1}_v]\Pr(v)} = (1+o(1))\frac{K_1^2}{K_2 L} \cdot \frac{A_{1,v}^2}{A_{0,v}A_{2,v}} = (1+o(1))\frac{K_1^2}{K_2 L} \prod_{2<p\le y} \frac{U_{2,p}}{U_{1,p}^2},$$

since the factors $r_p(v_p)$ and $(1 - v_p/\sqrt{p})$ cancel. The dependence on v has disappeared. Regrouping the convergent products,

$$\frac{K_1^2}{K_2} \prod_{2<p\le y} \frac{U_{2,p}}{U_{1,p}^2} = 12 \prod_{2<p\le y} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p>y} \left(1 - \frac{1}{p}\right)^{-1} \frac{U_{1,p}^2}{U_{2,p}}.$$

Since $(1-\frac{1}{p})U_{1,p} = 1 - \frac{1}{p(p+1)}$ and $(1-\frac{1}{p})^2 U_{2,p} = h(p) = 1 + \frac{(p-1)^2}{p^2(p+1)}$, the right side is $\kappa(y)$ as defined in (2). This proves the asymptotic in (c). For fixed y every pattern has $T(v) \le \sum_{p \le y} \log p \le (\log L)^3$ once X is large. Finally, the last product is $\prod_{p>y}(1 + O(p^{-2})) = 1 + O(1/y)$, and $\prod_{2<p\le y}(1-\frac{1}{p})^{-1} = (\frac{1}{2} + o(1))\mathrm{e}^{\gamma} \log y$ by Mertens' theorem, so $\kappa(y) = (6\mathrm{e}^{\gamma} + o(1)) \log y$ as $y \to \infty$. This proves (c).

For (b), by Cauchy–Schwarz in the finite space of pattern values,

$$C_X(y) = \sum_{v} \frac{\mathbb{E}[Z\mathbf{1}_v]^2}{\mathbb{E}[Z^2\mathbf{1}_v]},$$

attained by $M(v) = \mathbb{E}[Z\mathbf{1}_v]/\mathbb{E}[Z^2\mathbf{1}_v]$ (terms with zero denominator are omitted). Each term is at most $\Pr(v)$ by Cauchy–Schwarz. By Lemma 36 the patterns that are not good contribute $O(L^{-2})$. The good patterns contribute $(1+o(1))\frac{K_1^2}{K_2 L}\prod_{2<p\le y} U_{2,p}U_{1,p}^{-2} \cdot \Pr(\text{good}) = (1+o(1))\kappa(y)\Pr(\text{good})/L$, and

for $y = cL$ this is $(6\mathrm{e}^{\gamma} + o(1)) \log L/L$ by the computation above, since $\log(cL) = (1 + o(1)) \log L$.

The Euler product M_0. Expand $M_0^j = \sum_h b_j(h)\chi(h)$ for $j = 1, 2$, so $\sum_h b_j(h)a_j(h) = \prod_{2<p\le y} U_{j,p}^{-1}$ by the definition of the local laws. For each fixed k,

$$\sum_h |b_j(h)|a_j(h)(1 + \log h)^k \ll_k (1 + \log y)^k \sum_h |b_j(h)|a_j(h)h^{1/\log y} \ll (1 + \log y)^{O_k(1)},$$

since in the absolute Euler product both odd and positive even exponents contribute $O(1/p)$ at each $p \le y$, also with the factor $p^{2/\log y}$. The non-leading terms of the polynomials in Lemma 33, relative to $K_j a_j(h)$, are $O(L^{d_j-1}(1 + \log h)^{10})$ for $h \mid P^2$, where $d_1 = 1$, $d_2 = 3$; here we use $\Lambda_i \ge 0$ and $\sum_{n|h} \Lambda_i(n) = (\log h)^i$. The products $\prod U_{j,p}^{-1}$ are powers of $\log y$ up to bounded factors, so these terms are $o(L^{d_j} \prod_p U_{j,p}^{-1})$. The power errors are $O(X^{-1/4+O(c)+\varepsilon})$, since $\sum_h |b_j(h)| \le \prod_{p\le y}(1 + p^{-1/2})^j = X^{o(1)}$. Hence $\mathbb{E}[ZM_0] = K_1 L \prod U_{1,p}^{-1}(1+o(1))$ and $\mathbb{E}[Z^2 M_0^2] = K_2 L^3 \prod U_{2,p}^{-1}(1+o(1))$, and the ratio has the same asymptotic. □

Remark. By GRH [1] and the bound of Lamzouri, Li and Soundararajan for the least prime outside the kernel of a nonprincipal character [11], the values $\chi_{8m}(p)$ for $p \le C(\log X)^2$ determine m. At that scale the identity for $C_X(y)$ above gives $C_X(y) = \Pr(Z \ne 0)$ exactly, but the maximising M involves $1/Z$, so this says nothing new. It also shows that the uniformity in Theorem 7 cannot extend to $z \asymp (\log X)^2$: a centred linear statistic with Fourier support in $(-\frac{1}{4}, \frac{1}{4})$ has positive limiting variance by Theorem 7, while on singleton patterns its conditional variance is zero. This argument says nothing about the proportion in Theorem 2, since a singleton pattern may consist of a discriminant with $L(\frac{1}{2}, \chi_d) \ne 0$. The assertion of Theorem 2 that every pattern occurs fails for a simpler reason as soon as $z \ge (1 + \delta) \log X$ for a fixed $\delta > 0$: the pattern with $v_p = 0$ for every odd $p \le z$ requires $\prod_{2<p\le z} p$ to divide $m(d) \le 2X$, whereas $\prod_{2<p\le z} p = \mathrm{e}^{(1+o(1))z}$ by the prime number theorem.

A Programs, data and checks

The programs are available at `https://github.com/charlieoneill11/maths-releases`. They use only the Python standard library and exact

integer and rational arithmetic. Each program asserts the exact inequality of Proposition 32 for its stages. On a shared 2.1 GHz Xeon, stage 1 took about a minute and a half on four cores, and stages 2–5 took under two minutes each. On one core the program for stage 6 took about two and a half minutes, the program for stages 6–7 about five and a half minutes, and the last program, which carries out stages 6–8, about eight minutes. The programs for stages 6–8 start from the rational p_5, so they rely on the stage-5 program having passed; the last of them carries the running total through all the rows of (R7)–(R9) and checks $R_8 > p_8$. The spline arithmetic is implemented twice, in rational and in integer arithmetic. All eight stages and both cross-check programs of (ii) below were re-run on 24 November 2026 with Python 3.13, and stages 3 and 4 also with Python 3.9; all passed. The two cross-check programs took about half a minute and three and a half minutes.

Internal checks. (i) The shift formula of Lemma 28 is checked against an independent evaluation of $\mathcal{K}(\phi, T_a, T_b)$ from sine-kernel traces at four width pairs, and against a direct bounded-simplex inclusion–exclusion for four and five equal widths. (ii) Stages 2 and 3 evaluate Lemma 24 by enumerating partitions; stage 4 evaluates Lemma 25 by enumerating matchings. Stages 5–8 use the matching implementation of stage 4. Two programs recompute Gram entries by the other method and assert exact agreement. The first recomputes by enumerating labelled partitions all 208 Gram entries between a row of (R7) and a row of (R6) or (R7). The second recomputes all 4,287 inner products of stage 3 by enumerating matchings. By enumerating labelled partitions it recomputes all 8,385 entries among the rows of (R8) and (R9), all 2,838 entries between those rows and the rows of (R6)–(R7), and 3,000 distinct entries, chosen at random, between those rows and the rows of F_4. Both methods use the same routine for the individual cumulants $\mathcal{K}(\phi; S)$, so these checks do not replace an independent implementation. (iii) The integer spline arithmetic is compared with the rational one at several orders and points.

Quadratic rows. Table 7 lists the rows (R2): widths (a, b) in units of $\frac{1}{40}$ and the coefficients of $P_2, \dots, P_{10}$.

Table 7: Quadratic rows (R2): widths (a, b) in units of $\frac{1}{40}$ and the coefficients $c_1, \dots, c_5$ of $P_2, P_4, \dots, P_{10}$.

$40a$	$40b$	c_1	c_2	c_3	c_4	c_5
2	2	242110	545784	893667	883633	480446
2	4	399826	923502	1472691	1116727	55790

$40a$	$40b$	c_1	c_2	c_3	c_4	c_5
2	6	429521	873446	1226538	579765	−458315
2	8	490775	809395	916890	29337	−590695
2	10	489843	674629	634881	−330118	−532222
4	4	217218	449937	618872	293787	−293958
4	6	514063	890271	992554	154772	−919678
4	8	526481	757227	656167	−176942	−911451
4	10	526596	609495	400839	−373146	−680588
6	6	273666	408179	364064	−3445	−568669
6	8	552147	638418	341114	−129783	−982016
6	10	660272	548561	−63408	−432779	−383129
8	8	316143	252997	−99152	−155639	−227397
8	10	549018	429099	−404970	−659213	311778
10	10	81797	135236	−66936	−161016	61184
2	12	788703	501648	61582	−921403	−91304
2	16	404366	490165	334000	−947519	−342215
2	20	747200	703555	−372707	−565778	−196129
2	24	441560	395135	−380397	−298028	162976
2	28	392991	−89046	−241727	25331	23069
2	32	31459	−10862	−69852	62918	−35954
4	12	933394	460991	−332047	−1012033	401941
4	16	252131	210084	298855	−624900	−375488
4	20	724885	665993	−418774	−444113	−55054
4	24	444178	120995	−295143	−188998	155622
4	28	233882	−31352	−178523	36282	44244
4	32	−29032	−21044	1175	6480	4663
6	12	544089	417066	−190042	−961356	532361
6	16	433758	95376	−115720	−393861	−182619
6	20	587281	397771	−410872	−393042	163973
6	24	399738	42399	−381018	22024	91487
6	28	143749	−105449	−55640	52991	−15023
8	12	167988	165073	−162821	−404555	280858
8	16	544125	174395	−383810	−352024	62241
8	20	545526	95805	−367782	−211724	195705
8	24	255550	−9931	−271845	85180	44635
8	28	39090	−56179	4958	3250	3477
10	12	330379	148946	−458207	−63893	91533
10	16	389213	123909	−401577	−113210	58649
10	20	449703	−76163	−348853	21675	117961
10	24	215391	−120834	−94593	71968	484
12	12	187476	−16085	−360832	289597	−150378
12	16	203527	−92892	−69071	−138138	111003
12	20	295773	−116331	−211832	91469	31203
12	24	97850	−99964	−3197	7625	3344

$40a$	$40b$	c_1	c_2	c_3	c_4	c_5
14	16	152777	−98123	−94669	38153	11524
14	20	87181	−71263	−48929	36369	−7029

Rows (R8). Table 8 lists the rows (R8): widths in units of $\frac{1}{160}$ and the coefficients $c_1, \ldots, c_6$ of $P_2, \ldots, P_{12}$. The profile of each row is $10^{-9}\ell_{s,c}$ with $s = 1 - \sum A$.

Table 8: Rows (R8): widths in units of $\frac{1}{160}$ and the coefficients $c_1, \ldots, c_6$; blank entries are absent.

$160A$	c_1	c_2	c_3	c_4	c_5	c_6
∅	1749815	726190	−1588340	−347357	0	−9557
8	499022	−1296616	−4521807	−2080341	0	−6036267
16	−2612232	−3643729	4082174	4336558	0	1838789
24	−2050418	−4218237	422531	2884471	0	−5679359
32	1052539	282620	−2670391	1103750	0	−561546
40	−997931	−1762882	−1203356	−2197389	0	4677424
48	−103024	−2653974	−1781297	−2458696	0	1834261
56	−3338897	−2157820	70613	−527419	0	−1374624
64	438161	−3628713	−3823106	−1982383	0	−832315
72	−2916307	−406279	2679413	−4052298	0	1224721
80	−174475	−2606263	−3455414	2763980	0	−2233244
88	−1367930	96536	422475	−294443	0	−1311238
96	−1019779	−259309	−1020603	399824	0	−1390334
104	−1710048	−49009	1031901	−1695932	0	−953339
112	284691	801204	−331862	−630206	0	−698813
120	−378568	2208959	−1969551	966279	0	−247805
128	−1484157	1833101	−1848830	1605008	0	−194920
136	1266121	−2581888	1195917	144472	0	273212
144	4298	254373	581506	−410373	0	−139048
152	−562231	475507	173123	171793	0	−99721
8,8	−4988569	−7371431	−9508373			
8,16	−1283392	3283372	−3352633			
8,24	−342277	424261	−12314560			
8,32	−5309736	−3661433	5000713			
8,40	−5173031	1883946	13466438			
8,48	−35671562	8375311	51420543			
8,56	60253421	32482673	−32544235			
8,64	−11729527	−23649365	1432462			
8,72	608651	13111334	−3434638			
8,80	−30596123	−20423372	43407792			
8,88	35548323	33136759	−44929465			

$160A$	c_1	c_2	c_3	c_4	c_5	c_6
8,96	−26699318	−18768589	27044028			
8,104	13480473	10891380	−27313915			
8,112	−12187815	4133393	10413784			
8,120	13872953	−9859489	−532964			
16,16	−1348704	−7619986	16052846			
16,24	3246383	7029442	−4579900			
16,32	−3557565	5228042	9426199			
16,40	1001572	8056471	13438581			
16,48	−31202048	7441704	43055681			
16,56	48166270	30629941	−20975891			
16,64	−16757256	2050221	8229115			
16,72	37458703	24747679	−35668452			
16,80	−32854584	−30533389	36247056			
16,88	30653220	21642494	−35278486			
16,96	−24056984	1370938	12110092			
16,104	26740205	−11506013	−16658726			
16,112	495118	−8245978	16118833			
16,120	−136633	1596502	−2034265			
22,44	0	23243196				
24,24	−1608190	−8992825	9168698			
24,32	−10622380	−3312677	19474153			
24,40	−3666704	−6688686	19238657			
24,48	−4764365	−3413879	17047376			
24,56	17318746	20613999	9843991			
24,64	−15714917	10458472	9125947			
24,72	35074755	16731685	−29401254			
24,80	−28257196	−10649702	24032541			
24,88	24942117	4287734	−27777511			
24,96	−13196143	−7677739	19303779			
24,104	14917642	−10402076	−4679036			
24,112	−11538800	13536074	−2129386			
24,120	4668438	−3795300	817494			
32,32	−5477242	−1362023	13231721			
32,40	−18774572	6951759	38011329			
32,48	−18460024	7022205	8183674			
32,56	23883768	12658729	−12998383			
32,64	−25342874	−3524829	17396999			
32,72	23272903	4107748	−23486568			
32,80	−29194057	5724803	12635936			
32,88	21881703	−12236162	−15119598			
32,96	−12333084	146454	14649842			
32,104	792611	−1908595	−2845332			
32,112	−950393	3510452	−1120597			

$160A$	c_1	c_2	c_3	c_4	c_5	c_6
34,44	37380175					
38,46	28545901	−9432832				
40,40	−17771517	3923407	15246820			
40,44	10845935					
40,48	−19169407	21006591	−9273929			
40,56	24567649	7474208	−19478223			
40,64	−17560840	−3375201	14269435			
40,72	20988167	−8326206	−10968949			
40,80	−18819244	7929973	13008165			
40,88	14241553	−10355842	−6337157			
40,96	−12760643	8470353	4338291			
40,104	2381227	−4148164	1027100			
44,52	0	−10124428				
48,48	−4629908	7391220	7096778			
48,56	13630307	2469192	−6703782			
48,64	−8287722	4496266	4856693			
48,72	17923899	−14006427	−3730320			
48,80	−12482258	12772604	5359554			
48,88	8601461	−7428480	−1613661			
48,96	−4821259	6503340	86542			
56,56	6578896	−988602	−3439072			
56,64	−4385598	7583315	−3659058			
56,72	6949335	−7339399	−377397			
56,80	678154	1752422	2779176			
56,88	1478156	−2254075	33231			
64,64	2014277	727814	−3501727			
64,72	510311	−758993	−1711628			
64,80	2827531	−2225282	43515			
72,72	−534099	39559	−654864			

B Formal verification in Lean

The numerical certificate of Proposition 32 has also been proved in the Lean 4 proof assistant [10] with the Mathlib library [12]. So has Theorem 1, from GRH, two classical facts and Theorem 7 with the empty pattern, which enter as explicit hypotheses. The development uses version 4.29.1 of Lean and the matching release of Mathlib, and is available at `https://github.com/charlieoneill11/maths-releases`.

Statements. The formal statements define Fourier profiles, restricted cumulants and the model inner products of Section 4 as real numbers, and

the amplitude $F = F_8$ of Section 8 from its rows. Lean proves, with no hypothesis,

$$0.9453419890 < \frac{19 - \cot\frac{1}{4}}{16} + \langle G, F_8\rangle - \frac{1}{2}\langle F_8, F_8\rangle,$$

which is the consequence of Proposition 32 and Lemma 31 used in the proof of Theorem 1. The formal statements also define fundamental discriminants, the characters χ_d, the six classes (1) and $L(s, \chi_d)$ through Mathlib's Dirichlet L-functions, and state Theorem 1 for all discriminants and for each class. Lean proves these statements from three hypotheses.

- GRH for Dirichlet L-functions, as the proposition that every zero of Mathlib's $L(s, \chi)$ with $0 < \Re s < 1$ has $\Re s = \frac{1}{2}$. This is the statement proved in the formal project cited in [1]. That project uses a later version of Lean and Mathlib, so it is not imported here.
- The root number of $L(s, \chi_d)$ is 1 [2, Chs. 2, 6, 9], and the number of zeros with $T \le |\gamma| \le T + 1$ is $O(\log(|d|(T + 2)))$ uniformly in d and $T \ge 0$ [2, Ch. 16].
- Theorem 7 with the empty pattern, in each class.

The formal statements were reviewed against the paper and then kept unchanged. Theorems 2 and 3 are not formalised.

Proof. Every result of Sections 2 and 4 to 8 has a Lean counterpart. Lean proves Lemma 8 with the empty pattern, Theorem 22, the Wick-ordered evaluation of Section 6, the exact spline evaluation of Section 7 and Lemmas 30 and 31. The rows (R0)–(R9) are imported both from the published programs and from the TEX source of this paper, and Lean checks that the two agree. The eight stages of Proposition 32 are then evaluated by checkers written in Lean, each with a proof that an accepted run gives the stated real inequality. Their exact rational results agree with the published programs at every stage.

Trust. The checkers are evaluated with Lean's `native_decide` tactic, which runs compiled code, so accepting the formal proof means trusting the Lean compiler for these evaluations as well as the Lean kernel. The certificate and Theorem 1 use 41 such evaluations, which split the eight stages into blocks of Gram entries; the longest takes under five minutes. Apart from these evaluations, the proofs use only the three standard axioms of Lean: propositional extensionality, the axiom of choice and the soundness of quotients. They contain no unproved steps and declare no further axioms.

References

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