Six colours suffice for every unit-distance graph with at most 10,689 vertices or 245,510 edges

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Abstract

The chromatic number of the plane is known to be 6 or 7, so a finite unit-distance graph of chromatic number 7, if one exists, must be large. We prove that every unit-distance graph in the Euclidean plane with at most $10{,}689$ vertices is properly 6-colourable; the previous bound of this kind, due to Parts, covered graphs with at most $6{,}992$ vertices. We also prove that every finite unit-distance graph with at most $245{,}510$ edges is properly 6-colourable, whatever its number of vertices; the previous edge bound, $232{,}645$, follows from numerical estimates of de Grey and Parts. Equivalently, a seven-chromatic unit-distance graph, if one exists, has at least $10{,}690$ vertices and at least $245{,}511$ edges. Both results come from placing a fixed periodic six-colouring of the plane at random and then repairing what little goes wrong. For the edge bound we construct an everywhere-defined measurable periodic six-colouring in which a uniformly random unit segment is monochromatic with probability less than 4.07715×10^{-6}, and we show that a single monochromatic edge can be removed by recolouring one endpoint unless both endpoints see all six colours. A Fourier argument that is uniform over all stars shows that, at every vertex, the expected number of neighbours sharing its colour exceeds the probability that it sees all six colours by a fixed multiple of its degree. For the vertex bound we use a partial colouring with small blank holes. The cost of blank vertices is partly recovered by a forest of pairs of vertices that are blank together, chosen after the placement: nearby vertices, pairs whose difference is close to a vertical unit vector, and pairs of centres of two unit circles on each of which the same two common neighbours are consecutive. Crossing-number bounds control the degree variance of a minimal counterexample and force many such pairs of circles. Every numerical inequality is reduced to a finite computation in exact integer, rational or ball arithmetic, and the programs are published with the paper. The edge bound has also been proved in Lean 4 with Mathlib, and so has the vertex bound, from four published facts about crossing numbers taken as hypotheses.

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AI use and author contributions

The mathematical arguments and computer programs in this paper were developed by Astra (GPT-6.2 Astra, OpenAI). Fable (Fable 5.2, Anthropic) worked throughout as a second model: it proposed alternative approaches, reviewed arguments and raised objections. The manuscript was prepared from the project's research records with Claude (Anthropic). All computations were re-run before release, and the programs and certificates are available at https://github.com/charlieoneill11/maths-releases.

Open-weight models, Kimi K3.1 Pro (Moonshot AI) and DeepSeek V4.1 Flash (DeepSeek), were able to derive a significant proportion of these results when given guidance. We will report the details separately.

Charles O'Neill proposed the problem after the proof that the plane is not five-colourable [9] left six colours as the open case, and framed it through random placement of a periodic partial six-colouring: if the uncoloured fraction is ε, a random translate properly six-colours any configuration of fewer than $1/\varepsilon$ points. That reduction underlies both theorems. Max Kirkby set up and ran the experiments with open-weight models. Charles O'Neill has checked the arguments and takes full responsibility for the correctness of the results.

1 Introduction

The *chromatic number of the plane* $\chi(\mathbb{R}^2)$ is the least number of colours with which the points of the Euclidean plane can be coloured so that no two points at distance one receive the same colour. The hexagonal colouring of Isbell gives $\chi(\mathbb{R}^2) \leq 7$. De Grey [6] proved $\chi(\mathbb{R}^2) \geq 5$, and the exclusion of five colours [9] leaves $\chi(\mathbb{R}^2) \in \{6, 7\}$.

By the de Bruijn–Erdős theorem [5], $\chi(\mathbb{R}^2) = 7$ if and only if some finite unit-distance graph has chromatic number 7. It is therefore natural to ask how large such a graph must be. All bounds below are inclusive: a bound of N vertices means that every unit-distance graph with at most N vertices is 6-colourable, so that a seven-chromatic one has at least $N+1$ vertices; the same convention applies to edges. Pritikin [13] proved that every unit-distance graph with at most $6{,}197$ vertices is 6-colourable. Parts [12] constructed a periodic partial six-colouring of the plane with no monochromatic unit pairs whose uncoloured part has density $1/6992.1655\ldots$; a random translation then colours any $6{,}992$ given points properly, and Parts states the resulting bound of $6{,}992$ vertices. This is the vertex bound that Theorem 1 improves.

De Grey and Parts [7] considered instead complete colourings with a small probability of monochromatic unit segments, and obtained in this way the estimates that a seven-chromatic unit-distance graph has at least $232{,}646$ edges and at least $6{,}456$ vertices. The corresponding inclusive six-colour bounds are $232{,}645$ edges and $6{,}455$ vertices; the vertex bound is weaker than Parts's, and the edge bound is the one that Theorem 2 improves. De Grey and Parts describe these values as estimates: the relevant integrals were evaluated numerically with Mathematica, and no interval certificate is published there. We prove the following.

Theorem 1. *Every set of at most $10{,}689$ points in $\mathbb{R}^2$ has a colouring with six colours in which no two points at distance one have the same colour.*

Theorem 2. *Let G be a finite graph whose vertices are distinct points of $\mathbb{R}^2$ and whose edges are pairs of vertices at distance one. If G has at most $245{,}510$ edges then G is properly 6-colourable. There is no restriction on the number of vertices.*

Equivalently, a seven-chromatic finite unit-distance graph has at least $10{,}690$ vertices and at least $245{,}511$ edges. These are lower bounds for the size of a smallest seven-chromatic unit-distance graph, not estimates of it: whether such a graph exists at all is open. In particular, the failure of our final check at $10{,}690$ vertices (Section 10) only marks where this certificate stops. Theorem 2 rests on the following construction. For a measurable colouring c of $\mathbb{R}^2$ that is periodic under a lattice Λ, we say that a unit segment is *placed at random* if its direction is uniform on the circle and, independently, one endpoint is uniform on $\mathbb{R}^2/\Lambda$.

Theorem 3. *There is an everywhere-defined measurable six-colouring $c : \mathbb{R}^2 \to \{1, \dots, 6\}$, periodic under a lattice, such that the two endpoints of a randomly placed unit segment receive the same colour with probability at most*

$$p_E = \frac{2038573446989}{500000000000000000} = 4.077146893978 \times 10^{-6}.$$

By itself, Theorem 3 gives Theorem 2 for at most $245{,}269$ edges: rotate the graph uniformly and translate it uniformly modulo the period lattice, so that each edge becomes a randomly placed unit segment, and note that $245{,}269\, p_E < 1$. The last 241 edges come from a repair. If a placement has exactly one monochromatic edge, one of its endpoints can be recoloured unless both endpoints see all six colours among their neighbours (Lemma 20). We show that, for every vertex of degree d and whatever the directions of its d edges, the expected number of neighbours sharing its colour exceeds the

probability that it sees all six colours by at least $8 \cdot 10^{-9} d$ (Proposition 25). The proof bounds the angular Fourier coefficients of the colouring seen from a vertex and uses Fejér's kernel to show that no configuration of directions can make all of them small.

Theorem 1 needs more. The colouring used there leaves a set of density $q \approx 7.12 \times 10^{-5}$ uncoloured, and has monochromatic probability at most $p \approx 7.09 \times 10^{-7}$ (Proposition 12). A minimal counterexample on $10{,}689$ vertices has at most $403{,}078$ edges (Corollary 30). The union-bound estimate for the expected number of blank vertices and monochromatic edges is then at most $10{,}689\, q + 403{,}078\, p = 1.046268485\ldots$, which exceeds 1. So these two estimates alone do not prove the theorem. We recover the missing 5% in three ways. First, after the placement we choose a forest of pairs of blank vertices, and subtract the expected number of its edges (Lemmas 20 and 38). Its edges join nearby vertices, which tend to fall into the same hole; pairs of vertices whose difference is close to a vertical unit vector, which tend to fall into vertically adjacent holes; and pairs of centres of two unit circles on each of which the same two common neighbours are consecutive marked points; the distance between such centres is close to 2 (Section 8). Second, the same recolouring credit as for Theorem 2 applies to the partial colouring. Third, crossing-number bounds control the degree variance of a minimal counterexample, which limits the loss from vertices that cannot serve as forest tails, and they force many such pairs of circles.

Organisation and verification

Section 2 defines the family of periodic *wave colourings* and reduces their blank density, their monochromatic probability and their double-blank probabilities to one-dimensional integrals of piecewise-linear functions. Section 3 proves the recolouring credit for stars. Section 4 proves Theorems 2 and 3. Section 5 bounds the number of edges and the degree variance of a minimal counterexample using crossing numbers. Section 6 proves packing lemmas. Section 7 constructs the forest of short and vertical edges, and Section 8 adds the edges between circle centres. Section 9 lists the certified constants and completes the proof of Theorem 1.

Every constant is defined by an explicit finite formula, which is evaluated in exact integer or rational arithmetic or, where transcendental functions enter (the quadrature for the complete wave in Section 4 and the Fourier enclosures in Section 3.2), in rigorous ball arithmetic, always with rounding in the safe direction. We state each formula so that the computation can be reimplemented from the paper. The computations are carried out by the pro-

grams listed in Appendix A. The final finite check has two implementations written separately. Appendix B describes a formal proof of both theorems in Lean, with an independent implementation of every computation it uses.

Notation

Throughout, $H = \frac{1}{2}$. For a measurable set $K \subset \mathbb{R}^2$ write $\mathrm{area}(K)$ for its Lebesgue measure, and for $v \in \mathbb{R}^2$ write $\gamma_K(v) = \mathrm{area}\big(K \cap (K+v)\big)$ for its covariogram. We write $[z]_+ = \max(z, 0)$. For a real number z we write $\lfloor z \rfloor_{18}$ and $\lceil z \rceil_{18}$ for z rounded down and up to a multiple of 10^{-18}. We use the rational bounds

$$\pi_{\mathrm{lo}} = \tfrac{333}{106} < \pi < \tfrac{355}{113} = \pi_{\mathrm{hi}}.$$

Points of the plane are written (u, y), with y the vertical coordinate.

2 Wave colourings

2.1 Definition

A *wave datum* is a triple (x, ℓ, r) in which $x \in (\frac{9}{10}, 1)$ is rational and $\ell, r : [0, H] \to \mathbb{R}$ are continuous piecewise-linear functions with finitely many rational knots, such that:

(W1) $\ell(H - y) = \ell(y)$ and $r(H - y) = r(y)$ for $0 \le y \le H$;

(W2) $r(y) + r(y + H/2) = 2x$ for $0 \le y \le H/2$;

(W3) ℓ is nonincreasing on $[0, H/2]$, and there is $y_0 < H/4$ with $\ell = 0$ on $[y_0, H/2]$;

(W4) $0 \le \ell < \frac{1}{5}$ and $\frac{4}{5} < r < 1$ on $[0, H]$.

By (W1) we may extend ℓ and r to H-periodic functions on $\mathbb{R}$, and then $\ell(-y) = \ell(y)$. Moreover (W2) holds for every $y \in \mathbb{R}$: for $y \in [H/2, H]$ apply (W2) at $y - H/2$ and use periodicity.

Definition 4 (Wave colouring)**.** Let (x, ℓ, r) be a wave datum. For $i \in \mathbb{Z}$, the *i-th column* is the strip $2xi \le u < 2x(i+1)$. For $z = (u, y)$ in the i-th column put $u' = u - 2xi$ and $y' = y - \frac{3}{2}Hi$. The colour $c(z)$ is:

- $(1, \lfloor y'/H \rfloor \bmod 3)$ if $y' \notin H\mathbb{Z}$ and $\ell(y') < u' < r(y')$;
- $(2, \lfloor (y'+H/2)/H \rfloor \bmod 3)$ if $y'+H/2 \notin H\mathbb{Z}$ and $\ell(y'+H/2) < 2x - u' < r(y'+H/2)$;
- *blank* otherwise.

By (W2), $2x - r(y' + H/2) = r(y')$, so the first case requires $u' < r(y')$ and the second requires $u' > r(y')$. The two cases are disjoint and c is well defined. The six colours are $\{1, 2\} \times \mathbb{Z}/3$.

Let $T = \{(u, y) : 0 < y < H,\ \ell(y) < u < r(y)\}$ be the open *tile*, $T^* = \{(-u, y) : (u, y) \in T\}$ its mirror image, and

$$\Lambda = \mathbb{Z}(2x, \tfrac{3}{2}H) + \mathbb{Z}(0, 3H), \qquad \Gamma = \mathbb{Z}(2x, \tfrac{3}{2}H) + \mathbb{Z}(0, H).$$

Let $O = \{(u, y) : |y| \le y_0,\ |u| \le \ell(y)\}$ be the *hole*.

Lemma 5 (Structure)**.** *Let c be the wave colouring of a wave datum.*

1. *c is Λ-periodic, and Λ has covolume $6xH$.*
2. *Each colour class is a disjoint union $\tau_\kappa + T_\kappa + \Lambda$ for some $\tau_\kappa \in \mathbb{R}^2$ and $T_\kappa \in \{T, T^*\}$; three colour classes use T and three use T^*.*
3. *The blank set is the disjoint union of the translates $\zeta + O$, $\zeta \in \Gamma$, together with a null set (a locally finite union of segments). The lattice Γ has covolume $2xH$ and contains $(0, 1)$.*

Proof. (1) Translation by $(2x, \frac{3}{2}H)$ maps column i to column $i + 1$ and leaves (u', y') unchanged, and translation by $(0, 3H)$ changes y' by $3H$, which changes neither $\ell(y')$, $r(y')$ nor the row indices modulo 3.

(2) In column i, the points of colour $(1, k)$ with $\lfloor y'/H \rfloor = k + 3j$ form the set $(2xi, \frac{3}{2}Hi + (k + 3j)H) + T$. As (i, j) ranges over $\mathbb{Z}^2$ the offsets $(2xi, \frac{3}{2}Hi + 3jH)$ range over Λ, so the colour class is $(0, kH) + T + \Lambda$. Similarly, writing $y' + H/2 = (k + 3j)H + y''$ and $u' = 2x - u''$ with $(u'', y'') \in T$, the class of $(2, k)$ is $(2x, kH - H/2) + T^* + \Lambda$. These translates are disjoint because c is a function.

(3) In column i the non-tile points with $y' \notin H\mathbb{Z}$, $y' + H/2 \notin H\mathbb{Z}$ and $u' \ne r(y')$ are those with $u' \le \ell(y')$ or $2x - u' \le \ell(y' + H/2)$. Near the boundary $u = 2x(i + 1)$ between columns i and $i + 1$, the first condition for column $i+1$ reads $u - 2x(i+1) \le \ell\big(y - \frac{3}{2}H(i+1)\big)$, and the second condition for column i reads $2x(i+1) - u \le \ell\big(y - \frac{3}{2}Hi + \frac{H}{2}\big)$. Since $-\frac{3}{2}H \equiv \frac{H}{2} \pmod H$ these involve the same function of y. By (W1) and (W3) this function equals $\ell(y - \eta)$ when $|y - \eta| \le y_0$ for some $\eta \in \frac{3}{2}H(i + 1) + H\mathbb{Z}$, and vanishes otherwise. Hence the blank set near this boundary is $\bigcup_\eta \big((2x(i + 1), \eta) + O\big)$ up to segments, and the points $(2xi, \eta)$ obtained in this way form Γ. The translates are disjoint since $2y_0 < H$ and $2 \max \ell < 2x$. The excluded points lie on the lines $y' \in H\mathbb{Z}$, $y' + H/2 \in H\mathbb{Z}$, the seams $u' = r(y')$ and the lines $u \in 2x\mathbb{Z}$, a locally finite union of segments. Finally $(0, 1) = 2(0, H) \in \Gamma$. □

Lemma 6 (Symmetries). *Let c be the wave colouring of a wave datum, and put $T_{\mathrm{low}} = \{(u,y) \in T : y < H/2\}$ and $T_{\mathrm{high}} = \{(u,y) \in T : y > H/2\}$. Let $\sigma_1(u,y) = (u, H-y)$, $\sigma_2(u,y) = (-u, y)$ and $\tau_k(u,y) = (u, y + kH)$.*

1. *For $\gamma \in \{\sigma_1, \sigma_2, \tau_1\}$, and hence for every γ in the group they generate, there is a permutation π_γ of the six colours such that $c(\gamma(w)) = \pi_\gamma(c(w))$ for every $w \in \mathbb{R}^2$, where we put $\pi_\gamma(\text{blank}) = \text{blank}$.*
2. *The twelve sets $\tau_k(T_{\mathrm{low}})$, $\tau_k(T_{\mathrm{high}})$, $\sigma_2\tau_k(T_{\mathrm{low}})$, $\sigma_2\tau_k(T_{\mathrm{high}})$ $(k = 0,1,2)$ are images of T_{low} under isometries in the group generated by $\sigma_1, \sigma_2, \tau_1$, and they are pairwise disjoint modulo Λ.*

Proof. By the proof of Lemma 5(2), the class of $(1,k)$ is exactly $(0,kH)+T+\Lambda$ and the class of $(2,k)$ is exactly $(2x, kH - \frac{H}{2}) + T^* + \Lambda$; the blank set is the complement of the union of the six classes. So it suffices to show that each of $\sigma_1, \sigma_2, \tau_1$ maps every class onto a class.

Let $R = \mathrm{diag}(1,-1)$ and $S = \mathrm{diag}(-1,1)$. Both preserve Λ, since $R(2x, \frac{3}{2}H) = (2x, \frac{3}{2}H) - (0,3H)$ and $S(2x, \frac{3}{2}H) = -(2x, \frac{3}{2}H) + (0,3H)$. By (W1), $(u,y) \in T$ if and only if $(u, H-y) \in T$, so $RT = T - (0,H)$, and likewise $RT^* = T^* - (0,H)$. Since $\sigma_1(w) = Rw + (0,H)$,

$$\sigma_1\big((0,kH) + T + \Lambda\big) = (0,-kH) + T + \Lambda,$$
$$\sigma_1\big((2x, kH - \tfrac{H}{2}) + T^* + \Lambda\big) = (2x, (1-k)H - \tfrac{H}{2}) + T^* + \Lambda,$$

which are the classes of $(1,-k)$ and $(2,1-k)$. Next, $\sigma_2\big((0,kH) + T + \Lambda\big) = (0,kH) + T^* + \Lambda$, and $(0,kH) - (2x, (k+2)H - \frac{H}{2}) = (-2x, -\frac{3}{2}H) \in \Lambda$, so this is the class of $(2,k+2)$. As σ_2 is an involution, it maps the class of $(2,k+2)$ onto that of $(1,k)$. Finally τ_1 maps the class of (i,k) onto that of $(i,k+1)$. This proves (1).

For (2), $\tau_k(T_{\mathrm{high}}) = \tau_k\sigma_1(T_{\mathrm{low}})$ up to the segment $y = H/2$; since T is open and symmetric about $y = H/2$, in fact $\sigma_1(T_{\mathrm{low}}) = T_{\mathrm{high}}$ exactly. The set $\tau_k(T) = (0,kH)+T$ lies in the class of $(1,k)$, and $\sigma_2\tau_k(T) = (0,kH)+T^*$ lies in the class of $(2,k+2)$. These six classes are distinct, hence disjoint, and each is invariant under Λ. Within one class, T_{low} and T_{high} are disjoint subsets of one tile, and distinct translates of a tile by Λ are disjoint by Lemma 5(2). So the twelve sets are pairwise disjoint modulo Λ. □

2.2 Random placement

Let c be a wave colouring and $X \subset \mathbb{R}^2$ finite. A *random placement* of X is the colouring $x \mapsto c(\rho_\vartheta x + \tau)$, where ϑ is uniform on $[0, 2\pi)$, τ is uniform on a fundamental domain of Λ, and the two are independent. A vertex is *blank*

if its colour is blank, and a pair $\{x, y\}$ is *monochromatic* if both colours are equal and not blank. For fixed ϑ, each point $\rho_\vartheta x + \tau$ is uniform on $\mathbb{R}^2/\Lambda$.

Definition 7. For a wave colouring let q be the density of the blank set, and for a unit vector e let $P(e)$ be the probability that z and $z + e$ receive the same non-blank colour when z is uniform on $\mathbb{R}^2/\Lambda$. Put $\bar{p} = \frac{1}{2\pi}\int_0^{2\pi} P(e_\theta)\,\mathrm{d}\theta$, where $e_\theta = (\cos\theta, \sin\theta)$.

So in a random placement each vertex is blank with probability q, and each unit pair is monochromatic with probability exactly $\bar{p}$.

Lemma 8. $q = \dfrac{1}{xH}\displaystyle\int_0^H \ell(y)\,\mathrm{d}y.$

Proof. By Lemma 5(3), $q = \operatorname{area}(O)/(2xH)$, and $\operatorname{area}(O) = \int_{-y_0}^{y_0} 2\ell(y)\,\mathrm{d}y = 2\int_0^H \ell$ by (W1) and (W3). □

2.3 Monochromatic probability

Write $\gamma = \gamma_T$ for the covariogram of the tile.

Lemma 9. $\gamma(v_1, v_2) = \gamma(-v_1, v_2) = \gamma(v_1, -v_2)$, *and* $\gamma(v) = 0$ *unless* $|v_1| < \max r$ *and* $|v_2| < H$.

Proof. Always $\gamma(-v) = \gamma(v)$. The tile is invariant under $(u, y) \mapsto (u, H - y)$ by (W1), and this map sends $T \cap (T + v)$ onto $T \cap \big(T + (v_1, -v_2)\big)$. The last claim holds because $T \subset (0, \max r) \times (0, H)$. □

Lemma 10. *For* $0 \le \theta \le \pi/2$,

$$P(e_\theta) = \frac{1}{xH}\Big[\gamma(\cos\theta, \sin\theta) + \gamma\big(2x - \cos\theta,\ |\tfrac{3}{4} - \sin\theta|\big)\Big],$$

and $\bar{p} = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} P(e_\theta)\,\mathrm{d}\theta.$

Proof. Let $C = \tau_\kappa + T_\kappa + \Lambda$ be a colour class. Unfolding the torus, $\Pr(z \in C,\ z + e \in C) = \frac{1}{6xH}\sum_{\lambda\in\Lambda} \gamma_{T_\kappa}(\lambda - e)$. Since $\gamma_{T^*}(v_1, v_2) = \gamma(-v_1, v_2) = \gamma(v)$ by Lemma 9, summing over the six classes gives $P(e) = \frac{1}{xH}\sum_{\lambda\in\Lambda}\gamma(e - \lambda)$.

Now let $e = e_\theta$ with $0 \le \theta \le \pi/2$, and $\lambda = (2xi, \frac{3}{2}Hi + 3Hj)$. By Lemma 9 a nonzero term needs $|\cos\theta - 2xi| < \max r < 1$. Since $2x > \frac{9}{5}$ this forces $i \in \{0, 1\}$. If $i = 0$ we need $|\sin\theta - \frac{3}{2}j| < \frac{1}{2}$, so $j = 0$. If $i = 1$ we need $|\sin\theta - \frac{3}{4} - \frac{3}{2}j| < \frac{1}{2}$, so $j = 0$. The two remaining terms are $\gamma(e_\theta)$

and $\gamma(\cos\theta - 2x, \sin\theta - \frac{3}{4})$, and the latter equals $\gamma(2x - \cos\theta, |\frac{3}{4} - \sin\theta|)$ by Lemma 9.

Finally, Λ is invariant under both coordinate reflections, since $(-2x, \frac{3}{2}H) = -(2x, \frac{3}{2}H) + (0, 3H)$. With Lemma 9 this shows that $P(e)$ is invariant under $e \mapsto (-e_1, e_2)$ and $e \mapsto (e_1, -e_2)$, so the average over the circle equals the average over the first quadrant. □

Lemma 11 (Enclosure of the tile covariogram)**.** *Let $D \geq D_0 \geq 0$ and $0 \leq \delta_0 \leq \delta \leq \delta_1$. If $\delta_0 \geq H$ then $\gamma(D, \delta) = 0$. Suppose now that $\delta_0 < H$, and define $\ell^\flat(y) = \min\{\ell(y - \delta') : \delta' \in [\delta_0, \delta_1],\ \delta' \leq y\}$ for $\delta_0 \leq y \leq H$. Then*

$$\ell^\flat(y) = \begin{cases} \ell(y - \delta_0), & y - \delta_0 \leq H/2, \\ \ell(H/2), & y - \delta_1 \leq H/2 \leq y - \delta_0, \\ \ell(y - \delta_1), & y - \delta_1 \geq H/2, \end{cases} \tag{1}$$

and

$$\gamma(D, \delta) \leq \int_{\delta_0}^{H} \Big[r(y) - \ell^\flat(y) - D_0 \Big]_+ \, \mathrm{d}y.$$

Proof. If $\delta \geq H$, in particular if $\delta_0 \geq H$, then $\gamma(D, \delta) = 0$ by Lemma 9. Otherwise, slicing horizontally, $\gamma(D, \delta) = \int_\delta^H \big|(\ell(y), r(y)) \cap (\ell(y-\delta) + D,\ r(y-\delta) + D)\big| \, \mathrm{d}y$. The intersection of two open intervals (a_1, b_1) and (a_2, b_2) has length at most $[b_1 - a_2]_+$, so the integrand is at most $[r(y) - \ell(y - \delta) - D]_+ \leq \big[r(y) - \ell^\flat(y) - D_0\big]_+$. Extending the range of integration to $[\delta_0, H]$ only adds a nonnegative quantity. Formula (1) holds because the argument $y - \delta'$ ranges over an interval contained in $[0, H]$, and ℓ is nonincreasing on $[0, H/2]$ and nondecreasing on $[H/2, H]$ by (W1) and (W3). □

The function $\ell^\flat$ is piecewise linear: it is $\ell|_{[0,H/2]}$ translated by δ_0, followed by a constant piece, followed by $\ell|_{[H/2,H]}$ translated by δ_1. Hence the integrand $r - \ell^\flat - D_0$ is linear between consecutive points of the union of the knots of r and of $\ell^\flat$ in $[\delta_0, H]$, and its positive part can be integrated exactly. If a linear function on an interval of length ω has endpoint values $z_0 \leq z_1$, then the integral of its positive part is 0 if $z_1 \leq 0$, $\omega(z_0 + z_1)/2$ if $z_0 \geq 0$, and $\omega z_1^2/(2(z_1 - z_0))$ otherwise. This is nondecreasing in z_0 and in z_1.

Proposition 12 (Certified upper bound for $\bar{p}$)**.** *Let $N \geq 1$ and put $\tau_j = j/N$. For $0 \leq j < N$ let*

$$c_j^- \leq \frac{1 - \tau_{j+1}^2}{1 + \tau_{j+1}^2}, \quad c_j^+ \geq \frac{1 - \tau_j^2}{1 + \tau_j^2}, \quad s_j^- \leq \frac{2\tau_j}{1 + \tau_j^2}, \quad s_j^+ \geq \frac{2\tau_{j+1}}{1 + \tau_{j+1}^2}$$

be these values rounded outward to multiples of 10^{-12}*. Let* A_j *be the sum of the bounds of Lemma 11 for the two parameter choices*

$$(D_0, \delta_0, \delta_1) = (c_j^-,\ s_j^-,\ s_j^+),$$
$$(D_0, \delta_0, \delta_1) = \big(2x - c_j^+,\ \max(0, \tfrac{3}{4} - s_j^+, s_j^- - \tfrac{3}{4}),\ \max(\tfrac{3}{4} - s_j^-, s_j^+ - \tfrac{3}{4})\big),$$

where the bound is 0 *if* $\delta_0 \geq H$ *and is otherwise the integral of Lemma 11, evaluated with the values of* r *and of* $-\ell^\flat$ *at the integration knots rounded up to multiples of* 10^{-12}*, and with each piece rounded up to a multiple of* 10^{-24}*. (In the second choice* $\delta_0 = \frac{3}{4} - s_j^+$ *can exceed* H*.) Then*

$$\bar{p} \leq \frac{2}{\pi' \, xH} \sum_{j=0}^{N-1} A_j \cdot \frac{2N}{N^2 + j^2}, \qquad \pi' = \tfrac{314159}{100000}.$$

Proof. If $\tan(\theta/2) \in [\tau_j, \tau_{j+1}]$ then $\cos\theta \in [c_j^-, c_j^+]$ and $\sin\theta \in [s_j^-, s_j^+]$, by monotonicity of $\tau \mapsto \frac{1-\tau^2}{1+\tau^2}$ and $\tau \mapsto \frac{2\tau}{1+\tau^2}$ on $[0,1]$. For such θ, the first offset in Lemma 10 has horizontal component $\cos\theta \geq c_j^-$ and vertical component in $[s_j^-, s_j^+]$; the second has horizontal component $2x - \cos\theta \geq 2x - c_j^+$ and vertical component $|\frac{3}{4} - \sin\theta|$ in the stated interval. By Lemma 11 and the monotonicity of the positive-part integrals, $xH\,P(e_\theta) \leq A_j$ throughout the bin. The bin has angular length $2(\arctan\tau_{j+1} - \arctan\tau_j) \leq 2/(N(1+\tau_j^2)) = 2N/(N^2+j^2)$. Summing and applying Lemma 10 with $1/\pi < 1/\pi'$ gives the claim. □

The bound of Proposition 12 is computed in integer arithmetic. Heights and widths are integers in units of 10^{-12}, the knots of the profiles are such integers, and A_j is an integer in units of 10^{-24}. Each term $A_j \cdot 2N/(N^2+j^2)$ is rounded up to an integer in these units. All intermediate quantities are below 10^{37} and are held in signed 128-bit integers. A term with $\delta_0 \geq H$, or with $D_0 \geq \max r$ (when the integrand vanishes because $\ell^\flat \geq 0$), is set to zero without integrating. For speed, the bins are processed in $4{,}096$ consecutive blocks: the enclosure is first evaluated with the parameters of a whole block, and if the result is zero, which certifies that the integrand vanishes on the whole block, the fine bins of the block are skipped.

2.4 A height-integral bound for complete waves

When $\ell \equiv 0$ there is a sharper way to bound $\bar{p}$: bound the overlap at each height, and integrate first over the angle.

Proposition 13 (Height integral). *Let $(x, 0, r)$ be a wave datum with $\max r \le \frac{99}{100}$. For $0 \le y \le H$ put*

$$u = \arcsin y, \qquad v = \arcsin(\tfrac{3}{4} - y),$$
$$a = \arccos r(y), \qquad b = \arccos\big(2x - r(y)\big),$$

and

$$F_1(y) = \begin{cases} \displaystyle\int_a^u \big(r(y) - \cos\theta\big)\,\mathrm{d}\theta, & u > a, \\ 0, & u \le a, \end{cases}$$

$$F_2(y) = \begin{cases} \displaystyle\int_v^b \big(\cos\theta - 2x + r(y)\big)\,\mathrm{d}\theta, & b > v, \\ 0, & b \le v. \end{cases}$$

Then

$$\bar{p} \le \frac{2}{\pi x H} \int_0^H \big(F_1(y) + F_2(y)\big)\,\mathrm{d}y.$$

Proof. The quantities are well defined: $0 \le y \le \frac{1}{2}$, $\frac{1}{4} \le \frac{3}{4} - y \le \frac{3}{4}$, and $2x - r(y) = r(y + H/2) \in (\frac{4}{5}, 1)$ by (W2) and (W4).

Let $D \ge 0$ and $\delta \ge 0$. If $\delta \ge H$ then $\gamma(D, \delta) = 0$ by Lemma 9. Otherwise, slicing horizontally as in the proof of Lemma 11 with $\ell = 0$,

$$\begin{aligned} \gamma(D, \delta) &= \int_\delta^H \big|(0, r(y)) \cap (D, D + r(y - \delta))\big|\,\mathrm{d}y \\ &\le \int_0^H \mathbf{1}[\delta \le y]\ [r(y) - D]_+\ \mathrm{d}y. \end{aligned}$$

Apply this to the two terms of Lemma 10. For $0 \le \theta \le \pi/2$ it gives

$$xH\,P(e_\theta) \le \int_0^H \Big(\mathbf{1}[\sin\theta \le y]\,[r(y) - \cos\theta]_+ + \mathbf{1}\big[|\tfrac{3}{4} - \sin\theta| \le y\big]\,[r(y) - 2x + \cos\theta]_+\Big)\,\mathrm{d}y.$$

Integrate over $\theta \in [0, \pi/2]$ and exchange the order of integration, which is allowed since the integrand is nonnegative. Fix y. On $[0, \pi/2]$ the sine increases and the cosine decreases. So $\sin\theta \le y$ and $\cos\theta < r(y)$ hold together exactly for $a < \theta \le u$, which gives $\int_0^{\pi/2} \mathbf{1}[\sin\theta \le y]\,[r(y) - \cos\theta]_+\,\mathrm{d}\theta = F_1(y)$. For the second term, $\cos\theta > 2x - r(y) > \frac{4}{5}$ forces $\sin\theta < \frac{3}{5} < \frac{3}{4}$, so

on the support of $[r(y) - 2x + \cos\theta]_+$ the condition $|\frac{3}{4} - \sin\theta| \le y$ reads $\theta \ge v$, and $\cos\theta > 2x - r(y)$ reads $\theta < b$. This gives $F_2(y)$. Therefore $\int_0^{\pi/2} P(e_\theta)\,\mathrm{d}\theta \le \frac{1}{xH}\int_0^H (F_1 + F_2)$, and Lemma 10 gives the claim. □

Evaluating the integrals in θ,

$$G_1(y) := \int_a^u \big(r - \cos\theta\big)\,\mathrm{d}\theta = r(u-a) - y + \sqrt{1-r^2},$$
$$G_2(y) := \int_v^b \big(\cos\theta - c\big)\,\mathrm{d}\theta = \sqrt{1-c^2} - (\tfrac{3}{4} - y) - c(b-v), \qquad c = 2x - r, \tag{2}$$

where $r = r(y)$, and we use these expressions for every y, whatever the order of the limits.

Lemma 14. *Under the hypotheses of Proposition 13, suppose further that $|r'| \le 1$ on every linear piece of r. Then:*

1. *$G_1 \ge F_1 \ge 0$ and $G_2 \ge F_2 \ge 0$ on $[0, H]$;*
2. *on every closed interval $I \subseteq [0, H]$ on which r is affine, $F_1 + F_2$ is continuously differentiable and its derivative is Lipschitz with constant 38.*

Proof. (1) If $u > a$ then $F_1 = G_1 = \int_a^u (r - \cos\theta)\,\mathrm{d}\theta \ge 0$, because $\cos\theta \le \cos a = r$ on $[a, u]$. If $u \le a$ then $F_1 = 0$ and $G_1 = \int_u^a (\cos\theta - r)\,\mathrm{d}\theta \ge 0$, because $\cos\theta \ge \cos a = r$ on $[u, a]$. Similarly, if $b > v$ then $F_2 = G_2 \ge 0$, and if $b \le v$ then $F_2 = 0$ and $G_2 = \int_b^v (c - \cos\theta)\,\mathrm{d}\theta \ge 0$, because $\cos\theta \le \cos b = c$ on $[b, v]$.

(2) On I write r' for the constant slope; then $c' = -r'$ by (W2). On I we have $0 \le y \le \frac{1}{2}$ and $\frac{4}{5} < r, c \le \frac{99}{100}$, so G_1 and G_2 are smooth. Differentiating (2), and using $a' = -r'/\sqrt{1-r^2}$, $b' = -c'/\sqrt{1-c^2}$, $u'\cos u = 1$ and $v'\cos v = -1$,

$$G_1' = r'(u-a) + (r - \cos u)\,u', \qquad G_2' = r'(b-v) - (\cos v - c)\,v',$$

and

$$G_1'' = r'(u' - a') + (r' + u'\sin u)\,u' + (r - \cos u)\,u'',$$
$$G_2'' = r'(b' - v') + (v'\sin v - r')\,v' - (\cos v - c)\,v''.$$

Here $u' = (1-y^2)^{-1/2} \le 2/\sqrt{3} < 2$, $u'' = y(1-y^2)^{-3/2} < 1$, $|v'| = (1 - (\frac{3}{4} - y)^2)^{-1/2} \le 4/\sqrt{7} < 2$, $|v''| \le \frac{3}{4}(\frac{7}{16})^{-3/2} < 3$, and $|a'|, |b'| \le (1 - 0.99^2)^{-1/2} < 8$. Since $|r'| \le 1$, $|r - \cos u| \le 1$ and $|\cos v - c| \le 1$, we get $|G_1''|, |G_2''| \le (2+8) + (1+2)\cdot 2 + 3 = 19$ on I.

Now $F_1 = G_1\mathbf{1}[u > a]$, and $u > a$ if and only if $y^2 + r^2 > 1$. On I the function $y^2 + r(y)^2 - 1$ is a quadratic polynomial with leading coefficient $1 + r'^2 > 0$, so I splits into at most three intervals on each of which F_1 is either G_1 or 0. At a point where $u = a$ we have $G_1 = 0$ and, since $\cos u = \cos a = r$, also $G_1' = 0$. Hence F_1 is continuously differentiable on I, and F_1' is continuous and piecewise Lipschitz with constant 19, hence Lipschitz with constant 19. The same argument applies to $F_2 = G_2\mathbf{1}[b > v]$, since $b > v$ if and only if $c^2 + (\frac{3}{4} - y)^2 < 1$, again a quadratic condition with leading coefficient $1 + r'^2$, and at $b = v$ we have $G_2 = 0$ and $G_2' = 0$. □

Lemma 15 (Midpoint rule)**.** *If f is continuously differentiable on $[\alpha, \beta]$ and f' is Lipschitz with constant Λ_0, then $\left| \int_\alpha^\beta f - (\beta - \alpha) f(\frac{\alpha+\beta}{2}) \right| \le \Lambda_0(\beta - \alpha)^3/24$.*

Proof. Write $\mu = \frac{\alpha+\beta}{2}$ and $f(y) = f(\mu) + f'(\mu)(y - \mu) + \rho(y)$ with $|\rho(y)| \le \Lambda_0(y - \mu)^2/2$. The linear term integrates to zero, and $\int_\alpha^\beta \Lambda_0(y - \mu)^2/2 \,\mathrm{d}y = \Lambda_0(\beta - \alpha)^3/24$. □

Corollary 16. *Under the hypotheses of Lemma 14, suppose that every knot of r is a multiple of H/N for an integer N, and let $y_i = (i + \frac{1}{2})H/N$ for $0 \le i < N$. Let $\tilde{F}_i$ be any number with $\tilde{F}_i \ge F_1(y_i) + F_2(y_i)$. Then, with $\omega = H/N$,*

$$\bar{p} \le \frac{2}{\pi x}\Big(\frac{1}{N}\sum_{i=0}^{N-1} \tilde{F}_i + \frac{64\,\omega^2}{24}\Big).$$

Proof. Each cell $[i\omega, (i + 1)\omega]$ lies in an interval on which r is affine. Apply Lemma 15 with $\Lambda_0 = 38 \le 64$ on each cell, sum over the cells, divide by H, and use Proposition 13. □

By Lemma 14(1), we may take $\tilde{F}_i = \tilde{G}_{1,i} + \tilde{G}_{2,i}$, where $\tilde{G}_{1,i} = 0$ if a ball computation certifies $y_i^2 + r(y_i)^2 < 1$, and otherwise $\tilde{G}_{1,i}$ is the upper endpoint of a ball containing $G_1(y_i)$; similarly $\tilde{G}_{2,i} = 0$ if $c^2 + (\frac{3}{4} - y_i)^2 > 1$ is certified, and otherwise it is an upper bound for $G_2(y_i)$. So the comparisons that decide the cases never need to be resolved exactly.

2.5 Double-blank probability and the hole covariogram

Write $\kappa = \gamma_O$ for the covariogram of the hole, and $\hat{\ell}(y) = \ell(y)$ for $|y| \le y_0$, so that $O = \{|y| \le y_0,\ |u| \le \hat{\ell}(y)\}$.

Lemma 17. *1. $\kappa(v_1, v_2)$ is even in each coordinate and nonincreasing in $|v_1|$ and in $|v_2|$.*

2. *For $X \geq 0$ and $0 \leq Y \leq 2y_0$,*

$$\kappa(X,Y) = \int_{Y-y_0}^{y_0} \Big[\min\big(2\hat{\ell}(y),\ 2\hat{\ell}(y-Y),\ \hat{\ell}(y)+\hat{\ell}(y-Y)-X\big)\Big]_+ \,\mathrm{d}y,$$

and $\kappa(X,Y) = 0$ for $X \geq 0$ and $Y > 2y_0$.

3. *If z is uniform on $\mathbb{R}^2/\Lambda$, then for every $v \in \mathbb{R}^2$*

$$\Pr\big(z \text{ and } z+v \text{ are both blank}\big) \geq \frac{1}{2xH} \max\Big(\kappa(v),\ \kappa\big(v-(0,1)\big)\Big).$$

Proof. (1) The horizontal slice of O at height y is $[-\hat{\ell}(y), \hat{\ell}(y)]$, an interval centred at 0. For two intervals of half-lengths a, b centred at 0 and at X, the length of the intersection is $[\min(2a, 2b, a+b-|X|)]_+$, which is even and nonincreasing in $|X|$. Integrating over y gives the claims for v_1. Since $\hat{\ell}$ is even and nonincreasing in $|y|$ by (W1) and (W3), the vertical slice of O at abscissa u is also an interval centred at 0, and the same argument applies to v_2.

(2) This is the slice formula used in (1). The slices of O and of $O+(X,Y)$ at height y are both nonempty only if $|y| \leq y_0$ and $|y-Y| \leq y_0$, that is, for $Y \geq 0$, only if $Y - y_0 \leq y \leq y_0$. For $Y > 2y_0$ this range is empty and the overlap is zero.

(3) By Lemma 5(3), the events $\{z \in \zeta + O,\ z+v \in \zeta + O\}$ for $\zeta \in \Gamma$ are disjoint, and each implies that both points are blank. Their total probability is $\kappa(v)/(2xH)$. For the second bound use the events $\{z \in \zeta + O,\ z+v \in \zeta + (0,1) + O\}$, which have total probability $\kappa(v-(0,1))/(2xH)$ because $(0,1) \in \Gamma$. □

Definition 18 (Lower enclosure Ω)**.** For $X, Y \geq 0$ define $\Omega(X,Y) \geq 0$ as follows. Round X and Y up to multiples of 10^{-12}. If now $Y \geq 2y_0$ or $X \geq 2\hat{\ell}(0)$, put $\Omega(X,Y) = 0$. Otherwise partition the nonempty interval $[Y - y_0, y_0]$ at the knots of $\hat{\ell}$ and of $\hat{\ell}(\cdot - Y)$. At each partition point evaluate $\hat{\ell}$ and $\hat{\ell}(\cdot - Y)$ rounded down to multiples of 10^{-12}, and from these the value of $\min(2a, 2b, a+b-X)$. On each piece integrate the positive part of the linear function with these endpoint values, rounding down. Divide the sum by $2xH$.

Lemma 19. *If $|X'| \leq X$ and $|Y'| \leq Y$, then $\kappa(X',Y')/(2xH) \geq \Omega(X,Y)$.*

Proof. By Lemma 17(1) it suffices to take (X',Y') equal to (X,Y) rounded up. If $\Omega(X,Y) = 0$ there is nothing to prove, since $\kappa \geq 0$. Otherwise, on each piece, $\hat{\ell}$ and $\hat{\ell}(\cdot - Y)$ are linear, so the minimum of the three affine

functions in Lemma 17(2) is concave and lies above its chord. The positive part is monotone, and the chord computed from rounded-down endpoint values lies below the exact chord. □

3 Locked stars

In this section c is either the wave colouring of a wave datum or the complete colouring constructed in the proof of Theorem 3, and G is a finite graph whose vertices are points of $\mathbb{R}^2$ and whose edges are pairs of vertices at distance one. We place G at random as in Section 2. For a placement, a vertex v and its colour κ_v, let

- C_v be the number of neighbours of v with colour κ_v, if κ_v is not blank, and $C_v = 0$ if v is blank;
- $L_v = 1$ if v is not blank and its neighbours carry all six colours, and $L_v = 0$ otherwise;
- X be the number of *monochromatic* edges, whose ends have the same non-blank colour.

Each monochromatic edge is counted twice in $\sum_v C_v$, so $\sum_v C_v = 2X$. If $L_v = 1$ then $C_v \geq 1$, so $C_v - L_v \geq 0$ always.

Lemma 20 (One bad edge)**.** *Suppose that G has no proper 6-colouring. Fix a placement and let F be any forest on the vertex set of G. Let Y be the number of blank vertices minus the number of edges of F whose two ends are blank. Then $Y \geq 0$, and*

$$1 \leq Y + \frac{X}{2} + \frac{1}{4}\sum_v L_v.$$

Proof. The blank vertices and the edges of F joining two of them form a forest, and Y is its number of components. So $Y \geq 0$, and $Y \geq 1$ if some vertex is blank. Suppose no vertex is blank. If $X = 0$ the placement is a proper 6-colouring, which is impossible. If $X \geq 2$ the claim holds. If $X = 1$, let vw be the monochromatic edge. If $L_v = 0$, some colour κ is carried by no neighbour of v; in particular κ differs from the colour of w, which is the colour of v. Recolouring v with κ makes every edge at v proper and changes no other edge, so it gives a proper 6-colouring, which is impossible. Hence $L_v = 1$, and likewise $L_w = 1$, so $\frac{X}{2} + \frac{1}{4}\sum_u L_u \geq \frac{1}{2} + \frac{1}{2}$. □

Since $X = \frac{1}{2}\sum_v C_v$, taking expectations in Lemma 20 gives

$$1 \le \mathbb{E}Y + \mathbb{E}X - \frac{1}{4}\sum_v \mathbb{E}\,(C_v - L_v). \tag{3}$$

For a vertex v of degree $d \ge 1$ write its neighbours as $v + e_{\varphi_1}, \dots, v + e_{\varphi_d}$, and put

$$s_k = \frac{1}{d}\sum_{j=1}^{d} e^{ik\varphi_j} \qquad (k \in \mathbb{Z}).$$

These *power sums* satisfy $|s_k| \le 1$ and $|s_{-k}| = |s_k|$, and $|s_k|$ is unchanged if all the φ_j are replaced by $\varphi_j + \varphi$ or by $\varphi - \varphi_j$. For a bounded measurable function g on $\mathbb{R}/2\pi\mathbb{Z}$ write $\hat{g}(k) = \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-ikt}\,\mathrm{d}t$.

The image of v in a random placement is $z = \rho_\vartheta v + \tau$, and its neighbours go to $z + e_{\vartheta+\varphi_j}$. For fixed ϑ the point z is uniform on $\mathbb{R}^2/\Lambda$, so (ϑ, z) is uniform on $[0, 2\pi) \times \mathbb{R}^2/\Lambda$. For $z \in \mathbb{R}^2$ let $A_v(z)$ be the average over $\vartheta \in [0, 2\pi)$ of $C_v - L_v$, computed from the colours $c(z)$ and $c(z + e_{\vartheta+\varphi_j})$. Then

$$\mathbb{E}\,(C_v - L_v) = \frac{1}{6xH}\int_{\mathbb{R}^2/\Lambda} A_v(z)\,\mathrm{d}z, \qquad A_v \ge 0. \tag{4}$$

Lemma 21 (Star bounds)**.** *Let v have degree $d \ge 1$, and let $z \in \mathbb{R}^2$ with $\kappa = c(z)$ not blank.*

1. *Let $b \ne \kappa$ be a colour, and let g be a bounded measurable function on $\mathbb{R}/2\pi\mathbb{Z}$ with $g(t) \le \mathbf{1}[c(z + e_t) = \kappa] - \mathbf{1}[c(z + e_t) = b]$ for almost every t. Then for every $k \ge 1$,*
$$A_v(z) \ge \frac{d}{2}\left[\hat{g}(0) + |\hat{g}(k)|\,|s_k|\right]_+.$$

2. *If $c(z + e_t) = \kappa$ for all t in a measurable set $\mathcal{S} \subseteq \mathbb{R}/2\pi\mathbb{Z}$ of measure $2\pi f$, then $A_v(z) \ge d\ [f - 1/d]_+$.*

Proof. (1) For fixed ϑ let N_b be the number of neighbours of colour b. We claim $C_v - L_v \ge [C_v - N_b]_+$. If $L_v = 0$ this holds since $C_v, N_b \ge 0$. If $L_v = 1$ then $N_b \ge 1$ and $C_v \ge 1$, so $C_v - 1 \ge \max(0, C_v - N_b)$. Next, for each j and almost every ϑ,

$$\mathbf{1}[c(z + e_{\vartheta+\varphi_j}) = \kappa] - \mathbf{1}[c(z + e_{\vartheta+\varphi_j}) = b] \ge g(\vartheta + \varphi_j),$$

so $C_v - N_b \geq D(\vartheta) := \sum_j g(\vartheta + \varphi_j)$ for almost every ϑ. Hence $A_v(z) \geq \frac{1}{2\pi}\int D_+ = \frac{1}{4\pi}\int (D + |D|)$. Now $\frac{1}{2\pi}\int D = d\,\hat{g}(0)$, and

$$\frac{1}{2\pi}\int_0^{2\pi} |D(\vartheta)|\,\mathrm{d}\vartheta \geq \Big|\frac{1}{2\pi}\int_0^{2\pi} D(\vartheta)e^{-ik\vartheta}\,\mathrm{d}\vartheta\Big| = \Big|\sum_j e^{ik\varphi_j}\hat{g}(k)\Big| = d\,|s_k|\,|\hat{g}(k)|.$$

So $A_v(z) \geq \frac{d}{2}(\hat{g}(0) + |\hat{g}(k)||s_k|)$, and $A_v(z) \geq 0$.

(2) The number C_v is at least the number of j with $\vartheta + \varphi_j \in \mathcal{S}$, whose average over ϑ is df, and $L_v \leq 1$. So $A_v(z) \geq df - 1$, and $A_v(z) \geq 0$. □

Lemma 22 (Fejér)**.** *For all real $\varphi_1, \ldots, \varphi_d$ and every $K \geq 1$,*

$$\sum_{k=1}^{K} 2\Big(1 - \frac{k}{K+1}\Big)|s_k|^2 \geq \frac{K+1}{d} - 1.$$

Proof. The Fejér kernel $\Phi_K(t) = \frac{1}{K+1}\big|\sum_{j=0}^{K} e^{ijt}\big|^2 = \sum_{|k|\leq K}\big(1 - \frac{|k|}{K+1}\big)e^{ikt}$ is nonnegative and $\Phi_K(0) = K+1$. Hence

$$\sum_{|k|\leq K}\Big(1 - \frac{|k|}{K+1}\Big)|s_k|^2 = \frac{1}{d^2}\sum_{i,j=1}^{d}\Phi_K(\varphi_i - \varphi_j) \geq \frac{1}{d^2}\cdot d(K+1),$$

keeping only the diagonal terms. The left side is $1 + \sum_{k=1}^{K} 2(1 - \frac{k}{K+1})|s_k|^2$. □

3.1 Source boxes

Let T_{low} be as in Lemma 6; it lies in the tile T, of colour $(1,0)$. A *Fourier box* is a closed rectangle $\mathfrak{b} \subset \overline{T_{\mathrm{low}}}$ together with a colour $b \neq (1,0)$ and numbers $\mu_{\mathfrak{b}}$ and $h_{\mathfrak{b},k} \geq 0$ $(1 \leq k \leq K)$ such that for almost every $z \in \mathfrak{b}$ there is a function g as in Lemma 21(1) with $\hat{g}(0) \geq \mu_{\mathfrak{b}}$ and $|\hat{g}(k)| \geq h_{\mathfrak{b},k}$ for all $k \leq K$. A *native box* is a closed rectangle $\mathfrak{b} \subset \overline{T_{\mathrm{low}}}$ with a number $f_{\mathfrak{b}}$ such that for almost every $z \in \mathfrak{b}$ there is a set $\mathcal{S}$ as in Lemma 21(2) with $|\mathcal{S}| \geq 2\pi f_{\mathfrak{b}}$.

Lemma 23. *Let $\mathcal{F}$ be a family of Fourier boxes and $\mathcal{N}$ a family of native boxes, all with pairwise disjoint interiors. Let v be a vertex of degree $d \geq 1$ with power sums s_k. Then for every $1 \leq k \leq K$,*

$$\frac{\mathbb{E}\,(C_v - L_v)}{d} \geq \sum_{\mathfrak{b}\in\mathcal{F}} \frac{\operatorname{area}(\mathfrak{b})}{xH}\,[\mu_{\mathfrak{b}} + h_{\mathfrak{b},k}|s_k|]_+ + \sum_{\mathfrak{b}\in\mathcal{N}} \frac{2\operatorname{area}(\mathfrak{b})}{xH}\,\big[f_{\mathfrak{b}} - \tfrac{1}{d}\big]_+ .$$

Proof. Suppose first that c is a wave colouring. Let γ be one of the twelve isometries of Lemma 6(2), write $\gamma(w) = Mw + \zeta$ with $M \in \{I, R, S, RS\}$, and let π be the permutation of colours with $c \circ \gamma = \pi \circ c$, from Lemma 6(1). Define $t \mapsto t'$ by $e_{t'} = M^{-1}e_t$; it is one of t, $-t$, $\pi - t$, $\pi + t$. Let $\mathfrak{b}$ be a Fourier box, $z \in \mathfrak{b}$ and g as in the definition, and put $z' = \gamma(z)$ and $g'(t) = g(t')$. Then $c(z') = \pi(1,0)$, $\gamma(z + e_{t'}) = z' + e_t$, and

$$g'(t) \le \mathbf{1}[c(z' + e_t) = \pi(1,0)] - \mathbf{1}[c(z' + e_t) = \pi(b)]$$

for almost every t. Also $\hat{g}'(0) = \hat{g}(0)$ and $|\hat{g}'(k)| = |\hat{g}(k)|$. So Lemma 21(1) gives $A_v(z') \ge \frac{d}{2}\left[\mu_{\mathfrak{b}} + h_{\mathfrak{b},k}|s_k|\right]_+$ for almost every $z' \in \gamma(\mathfrak{b})$. In the same way Lemma 21(2) gives $A_v(z') \ge d\left[f_{\mathfrak{b}} - 1/d\right]_+$ for almost every $z' \in \gamma(\mathfrak{b})$ when $\mathfrak{b}$ is a native box. By Lemma 6(2), the sets $\gamma(\mathfrak{b})$, for the twelve isometries and all boxes, have pairwise disjoint interiors modulo Λ. Insert these bounds into (4), using $A_v \ge 0$ elsewhere: each box contributes $12\operatorname{area}(\mathfrak{b})/(6xH)$ times its bound.

If c is the complete colouring of Theorem 3, it agrees with the wave colouring c_0 of W_E off a locally finite union $\mathcal{N}_0$ of segments. A unit circle meets $\mathcal{N}_0$ in a finite set. So for every $z \notin \mathcal{N}_0$, the colours $c(z + e_t)$ and $c_0(z + e_t)$ agree for almost every t, and $A_v(z)$ is the same for c and c_0. Hence the bound for c_0 applies. □

Lemma 24 (Threshold test). *Let $\mathcal{F}, \mathcal{N}$ be as in Lemma 23, let $d \ge 1$ and $\beta' > 0$, and put $\mathcal{Z} = \sum_{\mathfrak{b}\in\mathcal{N}} \frac{2\operatorname{area}(\mathfrak{b})}{xH}\left[f_{\mathfrak{b}} - 1/d\right]_+$. For $1 \le k \le K$ let $t_k \in [0,1]$ be such that either $t_k = 1$ or*

$$\mathcal{Z} + \sum_{\mathfrak{b}\in\mathcal{F}} \frac{\operatorname{area}(\mathfrak{b})}{xH}\left[\mu_{\mathfrak{b}} + h_{\mathfrak{b},k}t_k\right]_+ \ge \beta'.$$

If $\mathcal{Z} \ge \beta'$, or if $\sum_{k=1}^{K} 2(1 - \frac{k}{K+1})t_k^2 < \frac{K+1}{d} - 1$, then $\mathbb{E}(C_v - L_v) \ge \beta' d$ for every vertex v of degree d.

Proof. If $\mathcal{Z} \ge \beta'$, use Lemma 23 with $\mathcal{F}$ empty. Otherwise suppose that $\mathbb{E}(C_v - L_v) < \beta' d$. The bound of Lemma 23 is nondecreasing in $|s_k|$, so $|s_k| < t_k$ whenever $t_k < 1$, and $|s_k| \le 1 = t_k$ otherwise. Then $\sum_k 2(1 - \frac{k}{K+1})|s_k|^2 \le \sum_k 2(1 - \frac{k}{K+1})t_k^2 < \frac{K+1}{d} - 1$, contradicting Lemma 22. □

Proposition 25 (Locked stars). *Let $\beta_E = 8 \cdot 10^{-9}$ and $\beta_V = 901 \cdot 10^{-12}$. In a random placement in the complete colouring c of Theorem 3, every vertex v of every finite unit-distance graph satisfies $\mathbb{E}\,(C_v - L_v) \ge \beta_E \deg v$. In a random placement in the wave colouring of the datum W_V of Section 9, every vertex satisfies $\mathbb{E}\,(C_v - L_v) \ge \beta_V \deg v$.*

The proof is a finite computation, described in the rest of this section, which verifies the hypotheses of Lemma 24 for every degree. For $d = 0$ there is nothing to prove.

3.2 Angular enclosures

Let $N_a = 2^{17}$. For $q \in \{0,1,2,3\}$ and $0 \le i \le N_a$ put $\alpha_{q,i} = q\pi/2 + 2\arctan(i/N_a)$, so that

$$e_{\alpha_{0,i}} = \Big(\frac{N_a^2 - i^2}{N_a^2 + i^2},\ \frac{2N_a i}{N_a^2 + i^2}\Big)$$

and $e_{\alpha_{q,i}}$ is obtained by rotating this through $q\pi/2$. For $i < j$, both coordinates of e_t are monotone on $[\alpha_{q,i}, \alpha_{q,j}]$, so e_t lies in the rectangle with opposite corners $e_{\alpha_{q,i}}$ and $e_{\alpha_{q,j}}$. If z lies in a box $\mathfrak{b}$, then $z + e_t$ lies in the *target rectangle* $\mathfrak{b} + [e_{\alpha_{q,i}}, e_{\alpha_{q,j}}]$, which the programs enlarge outward to coordinates that are multiples of 10^{-12}.

For a rectangle Q and a colour κ the programs compute a status: *in* only if Q lies in the closure of a single tile of colour κ, and *out* only if Q meets no tile of colour κ. They enumerate the columns and rows of Definition 4 that meet Q, and compare Q with lower and upper bounds for ℓ and r on the range of heights of Q in each row. Both profiles used below have ℓ nonincreasing and r nondecreasing on $[0, H/2]$, which the programs verify, so by (W1) the extreme values of ℓ and r on an interval of heights are attained at its endpoints or at $H/2$. Values of ℓ and r between knots are rounded outward to multiples of 10^{-12}.

Fix a box $\mathfrak{b}$ in T_{low} and an alternate colour b. In each quadrant, the programs classify index intervals $[i, j]$, starting from $[0, N_a]$, using the target rectangle. If the status of $(1,0)$ is *in*, the interval receives the value $+1$; otherwise, if the status of b is *in*, the value -1; otherwise, if both statuses are *out*, the value 0; otherwise the interval is bisected at $\lfloor (i+j)/2 \rfloor$, unless $j = i + 1$, in which case it is *terminal*. This produces a partition of the circle into arcs $I = [\alpha_{q,i}, \alpha_{q,j}]$.

Complete wave. Here $g_z(t) = \mathbf{1}[c(z + e_t) = (1,0)] - \mathbf{1}[c(z + e_t) = b]$ itself is used in Lemma 21(1). On a non-terminal arc with value ϵ we have $g_z = \epsilon$ for every $z \in \mathfrak{b}$ and almost every $t \in I$. On a terminal arc put $(m_I, \varrho_I) = (0,1)$ if neither status is *out*, $(\frac{1}{2}, \frac{1}{2})$ if only the status of b is *out*, and $(-\frac{1}{2}, \frac{1}{2})$ if only that of $(1,0)$ is *out*. On non-terminal arcs put $(m_I, \varrho_I) = (\epsilon, 0)$. Then $|g_z(t) - m(t)| \le \varrho(t)$ for every $z \in \mathfrak{b}$ and almost every t, where m and ϱ are the step functions with values m_I and ϱ_I. Hence

$$\hat{g}_z(0) \ge \hat{m}(0) - \hat{\varrho}(0), \qquad |\hat{g}_z(k)| \ge |\hat{m}(k)| - \hat{\varrho}(0),$$

and these right-hand sides, with $\hat{m}(k) = \frac{1}{2\pi}\sum_I m_I \frac{e^{-ik\alpha}-e^{-ik\alpha'}}{ik}$ for $I = [\alpha, \alpha']$, are computed in ball arithmetic with 100-bit precision. Their lower endpoints, rounded down to multiples of 10^{-10} (and replaced by 0 if negative, for h), are $\mu_{\mathfrak{b}}$ and $h_{\mathfrak{b},k}$.

Partial wave. Here the terminal arcs receive the value -1 if the status of b is not *out*, and 0 otherwise. The resulting step function g satisfies $g(t) \le \mathbf{1}[c(z+e_t) = (1,0)] - \mathbf{1}[c(z+e_t) = b]$ for every $z \in \mathfrak{b}$ and almost every t, and it does not depend on z. Its coefficients are computed in integer arithmetic with denominator $S = 10^{12}$. The length $2\arctan\frac{N_a}{N_a^2+i(i-1)}$ of the arc from $\alpha_{q,i-1}$ to $\alpha_{q,i}$ lies between $\lfloor 2S\eta\rfloor/S - 1/S$ and $\lceil 2S\eta\rceil/S$, where $\eta = N_a/(N_a^2 + i(i-1))$, since $0 \le 2\eta - 2\arctan\eta \le 2\eta^3/3 < 1/S$. Positive arcs use the lower bound and negative arcs the upper bound. For the Fourier coefficients, the k-th powers of the rounded points $e_{\alpha_{q,i}}$ are computed by repeated multiplication with truncation to $S^{-1}\mathbb{Z}[\mathrm{i}]$. Each truncation, and the initial rounding, costs less than $\sqrt{2}/S$, and if the k-th power is within $6k/S$ of $e^{\mathrm{i}k\alpha_{q,i}}$ then the next is within $(1+\sqrt{2}/S)6k/S + 2\sqrt{2}/S < 6(k+1)/S$; we use $k \le 512$. So each arc's term $(e^{ik\alpha'} - e^{ik\alpha})/k$ is computed with error at most $12/S$. The modulus is bounded below by an integer square root, the total error $12/S$ per arc is subtracted, and the results are divided by 2π using $3.1415926 < \pi < 3.1415927$ and rounded down to multiples of 10^{-10}.

Native boxes. Only the arcs with value $+1$ are used: $\mathcal{S}$ is their union, and $f_{\mathfrak{b}}$ is a lower bound for its measure divided by 2π, computed as above.

3.3 The certificate for the complete wave

All coordinates in this subsection are in units of 10^{-6}, and boxes are written $[u_0, u_1] \times [y_0, y_1]$. Put $r_y = \lfloor 10^6 r_E(10^{-6}y)\rfloor$. The Fourier boxes are:

- $[a', a''] \times [y, y+20]$ for $y = 0, 20, \ldots, 5980$, with alternate colour $(2,2)$, the colour of the tile across the left wall $u = 0$;
- $[u, u+20] \times [a', a'']$ for $u = 80, 100, \ldots, 5980$ and for $u = 866000, 866020, \ldots, 876380$, with alternate colour $(1,2)$, the colour of the tile below;
- $[r_y - a'', r_y - a'] \times [y, y+100]$ for $y = 0, 100, \ldots, 249900$, with alternate colour $(2,0)$, the colour of the tile across the seam $u = r(y)$;

where in the first two families (a', a'') runs over $(0,5)$, $(5,15)$, $(15,35)$, $(35,65)$, and in the third over $(0,10)$, $(10,30)$, $(30,60)$, $(60,95)$. There are $14{,}464$ of them. The three families are disjoint, since their abscissae lie in $[0,65]$, $[80, 876400]$ and $(876400, 10^6)$ respectively, and the program checks

that every box lies in $\overline{T_{\mathrm{low}}}$. The native boxes are the 252 squares $[u, u+50] \times [y, y+50]$ with $u \in \{0, 50, \dots, 550\}$ and $y \in \{0, 50, \dots, 1000\}$; they tile $[0, 600] \times [0, 1050]$. Exactly 14,148 Fourier boxes have interiors disjoint from this rectangle.

We use $K = 56$, $\beta' = \beta_E$, and thresholds t_k that are multiples of 10^{-5} found by bisection.

- For $d \le 23$ we take $\mathcal{N} = \emptyset$ and all 14,464 Fourier boxes. Then $\sum_k 2(1 - \frac{k}{57})t_k^2 = \frac{386575250753}{285000000000} < 1.357$, while $\frac{57}{d} - 1 \ge \frac{34}{23} > 1.478$.
- For $24 \le d \le 27$ we take the 252 native boxes and the 14,148 Fourier boxes disjoint from them. Then $\mathcal{Z} < \beta_E$, and the Fejér test passes with the following values.

d	$\mathcal{Z}$	$\sum_k 2(1 - \frac{k}{57})t_k^2$	$\frac{57}{d} - 1$
24	3.334×10^{-10}	1.36446	1.37500
25	1.715×10^{-9}	1.05445	1.28000
26	4.315×10^{-9}	0.56097	1.19231
27	7.676×10^{-9}	0.08319	1.11111

- For $d \ge 28$ we take $\mathcal{F} = \emptyset$ and the native boxes. At $d = 28$, $\mathcal{Z} = \frac{10356541361}{923677693607000000} > 1.12 \times 10^{-8} > \beta_E$, and $\mathcal{Z}$ is nondecreasing in d.

The program computes all these quantities exactly as rationals from the integers $\mu_{\mathfrak{b}}, h_{\mathfrak{b},k}, f_{\mathfrak{b}}$. This proves the first statement of Proposition 25.

3.4 The certificate for the partial wave

Again coordinates are in units of 10^{-6}. Put $r_y = \lfloor 10^6 r_V(10^{-6}y) \rfloor$ and $\ell_y = \lfloor 10^6 \ell_V(10^{-6}y) \rfloor + 1$. The Fourier boxes are:

- $[r_y - a'', r_y - a'] \times [y, y+10]$ for $y = 1, 11, \dots, 249981$ and (a', a'') in $(1, 4)$, $(4, 10)$, $(10, 20)$, $(20, 40)$, $(40, 70)$, with alternate colour $(2, 0)$;
- $[\ell_y + a', \ell_y + a''] \times [y, y+20]$ for $y = 9170, 9190, \dots, 17990$ and (a', a'') in $(0, 2)$, $(2, 5)$, $(5, 12)$, $(12, 25)$, $(25, 45)$, $(45, 80)$, $(80, 135)$, $(135, 200)$, with alternate colour $(2, 2)$;
- $[u, u+25] \times [a', a'']$ for $u = 4300, 4325, \dots, 10975$ and for $u = 861000, 861025, \dots, 874725$, and (a', a'') in $(1, 4)$, $(4, 10)$, $(10, 20)$, $(20, 40)$, $(40, 75)$, with alternate colour $(1, 2)$.

There are 132,621 of them. The native boxes are $[\ell_y + a', \ell_y + a''] \times [y, y+20]$ for $y = 5, 25, \dots, 9145$ and (a', a'') in $(0, 5)$, $(5, 15)$, $(15, 30)$, $(30, 55)$, $(55, 90)$,

$(90, 150)$, $(150, 240)$, $(240, 360)$, $(360, 540)$; there are $4{,}122$ of them, beside the hole. The program verifies that every box lies in T_{low} (strictly), and verifies by a sweep over the horizontal edges that all $136{,}743$ boxes have pairwise disjoint interiors.

We use all boxes, $\beta' = \beta_V$, thresholds that are multiples of 10^{-4}, and for each degree d the least $K = K_d \le 512$ for which the Fejér test of Lemma 24 passes. For instance $K_d = d$ for $d \le 8$, $K_{20} = 24$, $K_{40} = 71$, $K_{60} = 145$, the largest is $K_{79} = 309$, and $K_{88} = 100$. For $d \ge 89$ the native boxes alone give $\mathcal{Z} \ge \beta_V$, and $\mathcal{Z}$ is nondecreasing in d. The value 901 is the largest integer β for which the test passes at every degree with $\beta' = \beta \cdot 10^{-12}$. This proves the second statement of Proposition 25.

4 The complete colouring: proof of Theorems 2 and 3

Let $W_E = (x_E, 0, r_E)$ be the wave datum with

$$x_E = \frac{923677693607}{10^{12}}, \qquad \ell \equiv 0,$$

and r_E the piecewise-linear function with knots at the $2{,}049$ heights $kH/2048$, $0 \le k \le 2048$, whose values, integers in units of 10^{-12}, are published with the programs (Appendix A). Its values lie in $[0.876497, 0.970859]$. A program checks exactly that $r_E(H - y) = r_E(y)$ and $r_E(y + H/2) = 2x_E - r_E(y)$ at all knots, which gives (W1) and (W2) since both sides are piecewise linear with the same knots; that $\frac{4}{5} < r_E < \frac{99}{100}$ at all knots, which gives (W4) and the hypothesis of Proposition 13; that consecutive knot values differ by at most $H/2048$, so $|r_E'| \le 1$; and that r_E is nondecreasing on $[0, H/2]$. (W3) holds with $y_0 = 0$. Here the hole is a segment, so $q = 0$ and the blank set is null.

Proof of Theorem 3. Let c_0 be the wave colouring of W_E. Here $y_0 = 0$, so the hole O is the single point $(0, 0)$, and by Lemma 5(3) and its proof the blank set of c_0 is contained in the set $\mathcal{N}_0$ formed by the horizontal lines $y' \in H\mathbb{Z}$ and $y' + H/2 \in H\mathbb{Z}$, the vertical lines $u \in 2x\mathbb{Z}$, and the polygonal seams $u' = r_E(y')$; this is a locally finite union of segments. The blank set is a proper subset of $\mathcal{N}_0$: for instance the point $(\frac{3}{2}, 0)$ of column 0 lies on the line $y' = 0$ but receives the colour $(2, 0)$. The boundary of every open tile lies in $\mathcal{N}_0$, and the open tiles are disjoint and cover $\mathbb{R}^2 \setminus \mathcal{N}_0$, so every component of $\mathbb{R}^2 \setminus \mathcal{N}_0$ lies in a single open tile. Fix a unit vector ν that is neither horizontal, vertical, nor parallel to any of the finitely many segments of the seams. For $z \in \mathbb{R}^2$ the open segment $\{z + \varepsilon\nu : 0 < \varepsilon < \varepsilon_z\}$ avoids $\mathcal{N}_0$ for some $\varepsilon_z > 0$,

because only finitely many segments of $\mathcal{N}_0$ meet a neighbourhood of z, and each meets the line $z + \mathbb{R}\nu$ in at most one point. This open segment is connected, so it lies in one open tile. Let $c(z)$ be the colour of that tile. Then c is defined everywhere, agrees with c_0 off the null set $\mathcal{N}_0$, and is therefore measurable and Λ-periodic.

A randomly placed unit segment has an endpoint in $\mathcal{N}_0$ with probability zero, so its probability of being monochromatic under c equals $\bar{p}$ for c_0. We apply Corollary 16 with $N = 2048 \cdot 256 = 524{,}288$, so that $\omega = 2^{-20}$ and $64\omega^2/24 = 1/412316860416$. The program evaluates $\tilde{F}_i$ at the $524{,}288$ midpoints as described after Corollary 16, in ball arithmetic with 100-bit precision, and rounds the upper endpoint of the resulting ball for the right side of Corollary 16 up to a multiple of 10^{-18}. The result is p_E. $\square$

Proof of Theorem 2. Let G have m edges and suppose it has no proper 6-colouring. Place G at random in the complete colouring c of Theorem 3. No vertex is blank, so with $F = \emptyset$ we have $Y = 0$ in Lemma 20. Each edge is monochromatic with probability $\bar{p} \le p_E$, so $\mathbb{E}X \le m p_E$. By (3) and Proposition 25,

$$1 \le m p_E - \frac{1}{4}\sum_v \beta_E \deg v = m\Big(p_E - \frac{\beta_E}{2}\Big) = m \cdot \frac{2036573446989}{500000000000000000}.$$

But $245{,}510 \cdot 2036573446989 < 5 \cdot 10^{17}$, so $m > 245{,}510$. $\square$

Without Proposition 25, the same argument gives $1 \le m p_E$, and $245{,}269\, p_E < 1$. The computations of p_E and of Proposition 25 for c are reproduced by the programs of Appendix A.

Remark. The profile r_E was found as follows; the proof does not use this. At height y with $r = r_E(y)$, the integrand of Proposition 13 is $F_1(y) + F_2(y)$. By (W1) and (W2), the heights t, $H - t$, $H/2 - t$, $H/2 + t$ carry widths $r, r, 2x - r, 2x - r$, so for fixed x the objective decouples into convex one-dimensional problems, one for each $t \in [0, H/4]$. These were solved by bisection at the heights $t \in \frac{1}{4096}\mathbb{Z} \cap [0, \frac{1}{8}]$, the widths were rounded to multiples of 10^{-12}, and the symmetries (W1) and (W2) were imposed exactly.

5 Crossings and the degree variance

A *drawing* of a multigraph without loops represents vertices by distinct points of the plane and edges by Jordan arcs joining their endpoints and containing no other vertex. If any two edges of a drawing $\mathcal{D}$ have finitely

many common points, let $\mathrm{cr}(\mathcal{D})$ be the number of triples (e, f, z) with $\{e, f\}$ an unordered pair of distinct edges and z a common interior point of e and f. The crossing number $\mathrm{cr}(G)$ is the minimum number of crossings in a drawing of G. By local redrawing near each common point (removing touchings and separating points where several edges meet), $\mathrm{cr}(G) \le \mathrm{cr}(\mathcal{D})$ for every such drawing [14].

Theorem 26 (Büngener–Kaufmann; Büngener–Franz–Kaufmann–Pfister)**.** *Every simple graph G with $N \ge 3$ vertices and M edges satisfies* $\mathrm{cr}(G) \ge 6M - \frac{266}{9}(N-2)$.

Büngener and Kaufmann [3] prove that every graph with $N \ge 3$ vertices and M edges satisfies $\mathrm{cr}(G) \ge 5M - \frac{203}{9}(N-2)$; this is Theorem 4(b) of the journal version and Theorem 6(b) of arXiv:2409.01733v2. For $M \le 7(N-2)$ it implies Theorem 26, since the two right-hand sides agree at $M = 7(N-2)$. For $M > 7(N-2)$ we use the density theorem of Büngener, Franz, Kaufmann and Pfister [4, Theorem 7]: a simple graph on $N \ge 3$ vertices with a simple drawing in which every edge has at most five crossings has at most $7(N-2)$ edges. Fix a drawing of G with $\mathrm{cr}(G)$ crossings. It is simple, that is, adjacent edges do not cross and two edges cross at most once [14], and so are its subdrawings. Delete edges one at a time, each with at least six crossings in the current subdrawing, until at most $7(N-2)$ edges remain; this is possible by the density theorem. If M' edges remain, then $M' \le 7(N-2)$ and

$$\begin{aligned}\mathrm{cr}(G) &\ge 6(M - M') + 5M' - \tfrac{203}{9}(N-2)\\ &\ge 6M - 7(N-2) - \tfrac{203}{9}(N-2) = 6M - \tfrac{266}{9}(N-2).\end{aligned}$$

This is the argument in the proof of Theorem 17 of [4].

Lemma 27 (Sampling)**.** *Let G be a simple graph with n vertices and m edges, and let $4 \le k \le n$. Put $P_2 = \frac{(k)_2}{(n)_2}$ and $P_4 = \frac{(k)_4}{(n)_4}$, where $(a)_i = a(a-1)\cdots(a-i+1)$. Then*

$$\mathrm{cr}(G) \ge c_k(m) := \frac{6mP_2 - \frac{266}{9}(k-2)}{P_4}.$$

Proof. Fix a drawing of G with $\mathrm{cr}(G)$ crossings. In a drawing with the minimum number of crossings, adjacent edges do not cross [14], so each crossing involves four distinct vertices. Let S be a uniformly random k-subset of the vertices and keep the induced subdrawing. An edge survives with probability P_2 and a crossing survives with probability P_4. Since $k \ge 4$, $P_4 > 0$. Applying Theorem 26 to $G[S]$, which has $k \ge 3$ vertices, and taking expectations gives $P_4\,\mathrm{cr}(G) \ge 6mP_2 - \frac{266}{9}(k-2)$. □

Lemma 28 (Halving). *Let $\mathcal{H}$ be a multigraph without loops on $n \geq k \geq 4$ vertices, with $2m$ edges and every pair of vertices joined by at most two edges. Let $\mathcal{D}$ be a drawing of $\mathcal{H}$ in which any two edges have finitely many common points. Then* $\operatorname{cr}(\mathcal{D}) \geq 4c_k(m)$.

Proof. Let b be the number of pairs of parallel edges and a the number of remaining edges, so $a + 2b = 2m$. Form a random simple spanning subgraph G by keeping one edge of each parallel pair, chosen uniformly, and keeping each remaining edge independently with probability $\frac{1}{2}$. Every edge of $\mathcal{H}$ is kept with probability exactly $\frac{1}{2}$, edges from different classes are kept independently, and two edges of the same parallel pair are never both kept. Hence $\mathbb{E}[\operatorname{cr}(\mathcal{D}|_G)] \leq \frac{1}{4}\operatorname{cr}(\mathcal{D})$, where $\mathcal{D}|_G$ is the subdrawing. Also $\mathbb{E}[e(G)] = b + a/2 = m$. Since c_k is an affine function of m, Lemma 27 gives

$$\tfrac{1}{4}\operatorname{cr}(\mathcal{D}) \geq \mathbb{E}[\operatorname{cr}(\mathcal{D}|_G)] \geq \mathbb{E}[\operatorname{cr}(G)] \geq \mathbb{E}[c_k(e(G))] = c_k(m). \qquad \square$$

Lemma 29 (Degree variance). *Let $X \subset \mathbb{R}^2$ be a set of $n \geq k \geq 4$ points whose unit-distance graph has m edges and degrees d_v, and suppose that every non-isolated vertex has degree at least 3. Then*

$$\sum_{v \in X} \Big(d_v - \frac{2m}{n}\Big)^2 \leq 2V_k(m), \qquad V_k(m) = n(n-1) - 4c_k(m) - \frac{2m^2}{n} + m.$$

Proof. For each non-isolated v let S_v be the unit circle centred at v. The points of X on S_v are exactly the $d_v \geq 3$ neighbours of v, and they divide S_v into d_v arcs, each joining two distinct points of X. Let $\mathcal{H}$ be the multigraph on X whose edges are these $\sum_v d_v = 2m$ arcs, drawn as the arcs themselves; call this drawing $\mathcal{D}$. No arc contains a point of X in its interior. Two distinct points lie on at most two unit circles, and on a circle with at least three marked points at most one arc joins two given marked points, so $\mathcal{H}$ has edge multiplicity at most two.

Arcs of the same circle have disjoint interiors, and two distinct circles meet in at most two points. Count the pairs $(\{v, v'\}, z)$ with $v \neq v'$ non-isolated and $z \in S_v \cap S_{v'}$. There are at most $2\binom{n}{2} = n(n-1)$ of them. A point $u \in X$ lies on S_v if and only if v is a neighbour of u, so the pairs with $z \in X$ number $\sum_u \binom{d_u}{2}$. Each pair with $z \notin X$ contributes at most one crossing of $\mathcal{D}$, since z lies in the interior of exactly one arc of each of the two circles. Therefore $\operatorname{cr}(\mathcal{D}) \leq n(n-1) - \sum_u \binom{d_u}{2}$, and by Lemma 28,

$$\sum_u \binom{d_u}{2} \leq n(n-1) - 4c_k(m).$$

Since $\sum_u \binom{d_u}{2} = \frac{1}{2}\sum_u (d_u - 2m/n)^2 + 2m^2/n - m$, the lemma follows. $\square$

The use of the unit circles about the points to bound unit distances by the crossing lemma is due to Székely [15]. Subtracting the intersections at the points themselves is due to Ágoston and Pálvölgyi [1], who also give a multiplicity-two reduction of the kind proved in Lemma 28.

We use $k = 2090$ throughout. Then $4c_k(m) = \sigma m - \kappa_0$ with

$$\sigma = \frac{24(n-2)(n-3)}{(k-2)(k-3)}, \qquad \kappa_0 = \frac{4 \cdot 266\, n(n-1)(n-2)(n-3)}{9k(k-1)(k-3)},$$

so $V(m) := V_{2090}(m) = n(n-1) + \kappa_0 + (1-\sigma)m - 2m^2/n$ is a concave quadratic. Since $\sigma > 1$, V is strictly decreasing on $[0, \infty)$.

Corollary 30 (Edge ceiling)**.** *Let $n = 10{,}689$ and $m^* = 403{,}078$. Then $V(m^*) > 0 > V(m^* + 1)$. Consequently, under the hypotheses of Lemma 29, $m \le m^*$.*

Proof. For $n = 10{,}689$ we have $\sigma = 38067094/60523 \approx 628.97$, and exact evaluation gives $V(403{,}078) \approx 107.1$ and $V(403{,}079) \approx -671.7$. The left side of Lemma 29 is nonnegative, so $V(m) \ge 0$, and V is decreasing. □

6 Packing lemmas

In this section P is a finite set of points with pairwise distances greater than 2ϱ, for some $\varrho > 0$, and $t > 0$. Write $\nu_P(z,t) = \#\{z' \in P \setminus \{z\} : |z - z'| \le 2t\}$.

Lemma 31 (Pointwise bound)**.** *For every $z \in P$, $\nu_P(z,t) \le k_\varrho(t)$, where $k_\varrho(t) = 0$ if $t \le \varrho$ and otherwise*

$$k_\varrho(t) = \min\Big(\big\lfloor (1 + 2t/\varrho)^2 \big\rfloor - 1,\ \mathcal{A}\big(\lceil 100t/\varrho \rceil\big)\Big),$$

with $\mathcal{A}$ defined in the proof.

Proof. If $t \le \varrho$ the count is zero by separation. The open discs of radius ϱ about the points counted and about z are disjoint and lie in the disc of radius $2t + \varrho$ about z, which gives the first bound.

For the second, scale by $1/(2\varrho)$. The counted points then have pairwise distances greater than 1 and distances from z in $(1, \varpi]$ with $\varpi = t/\varrho$. For $1 \le \varpi_1 \le \varpi_2$ put $C(\varpi_1, \varpi_2) = \max \frac{a^2+b^2-1}{2ab}$ over $a, b \in \{\varpi_1, \varpi_2\}$. If two points at distances $a, b \in [\varpi_1, \varpi_2]$ from z subtend an angle ϕ at z, then $a^2 + b^2 - 2ab\cos\phi > 1$, so $\cos\phi < \frac{a^2+b^2-1}{2ab} \le C(\varpi_1, \varpi_2)$. The last inequality holds because $a \mapsto \frac{a^2+b^2-1}{2ab}$ is convex on $a > 0$ when $b \ge 1$, and symmetrically

in b, so its maximum over the square $[\varpi_1,\varpi_2]^2$ is attained at a corner. Let $N(\varpi_1,\varpi_2)$ be the least integer $k\in[4,127]$ with

$$1-\tfrac{\psi^2}{2}+\tfrac{\psi^4}{24}-\tfrac{\psi^6}{720}>C(\varpi_1,\varpi_2),\qquad \psi=\tfrac{44}{7(k+1)},$$

and $N=\infty$ if there is none or $C\ge 1$. Suppose $k+1$ points lie at distances in $[\varpi_1,\varpi_2]$. Two of them are angularly consecutive about z and subtend an angle $\phi\le 2\pi/(k+1)\le\psi\le\pi$, so $\cos\phi\ge\cos\psi\ge 1-\psi^2/2+\psi^4/24-\psi^6/720$, a contradiction. So at most $N(\varpi_1,\varpi_2)$ points lie in the annulus. Finally put

$$\mathcal{A}(M)=\min\Big(N(1,\tfrac{M}{100}),\ \min_{c}\big[N(1,\tfrac{c}{100})+N(\tfrac{c}{100},\tfrac{M}{100})\big]\Big),$$

the inner minimum over $c\in\{110,120,130,\dots\}$ with $c<M$. Since $\varpi\le\lceil 100t/\varrho\rceil/100$, every counted point lies in one of the annuli used. □

Lemma 32 (Average degree)**.** *Let G_t be the graph on P joining pairs at distance at most $2t$, and put $\varpi=t/\varrho$. If $\varpi^2\le 2$ then the degrees in G_t sum to at most $6|P|$. If $\varpi\le\frac{39}{25}$ then they sum to at most $8|P|$.*

Proof. Scale by $1/(2\varrho)$, so that points are more than 1 apart and edges have length at most ϖ. Consider the straight-line drawing of G_t. No edge passes through a third point, since that edge would then have length greater than 2.

Let $\varpi^2\le 2$, and suppose that edges pq and $p'q'$ cross. Their endpoints form a convex quadrilateral with diagonals pq and $p'q'$, and one of its angles is at least $\pi/2$, say the angle at p. By the cosine rule $|p'q'|^2\ge|pp'|^2+|pq'|^2>2$, a contradiction. So the drawing is plane, G_t is planar, and it has at most $3|P|-6$ edges if $|P|\ge 3$ and at most one edge otherwise. In both cases the degrees sum to at most $6|P|$.

Let $\varpi\le\frac{39}{25}$. By the first part we may assume $\varpi^2>2$. Put $h=\frac{5}{8}$. Place an edge on $[0,\varpi']\times\{0\}$ with $\varpi'\le\varpi$. Every other point (X,Y) is at distance more than 1 from both endpoints. If $0\le X\le\varpi'$ then $Y^2>1-(\varpi'/2)^2\ge 1-(39/50)^2>h^2$; otherwise its distance to the segment exceeds 1. So every other point has distance more than h from the segment. Suppose an edge pp' crosses the segment at o. Then $|po|\le\varpi-|op'|<\varpi-h<1$. If p projected outside $[0,\varpi']$, say $X_p<0$, then $|po|\ge|p|>1$; so p projects inside, its height satisfies $h<|Y_p|<\varpi-h$, and from $X_p^2+Y_p^2>1$ we get $X_p>(1-(\varpi-h)^2)^{1/2}>\frac{7}{20}$, and likewise $\varpi'-X_p>\frac{7}{20}$. Hence p lies in one of the two boxes $[\frac{7}{20},\varpi'-\frac{7}{20}]\times\pm[h,\varpi-h]$, each of diameter at most $\big((\varpi-\frac{7}{10})^2+(\varpi-\frac{5}{4})^2\big)^{1/2}<1$. Each box contains at most one point, and a

crossing edge has one endpoint in each box. So every edge is crossed by at most one other edge.

It follows that G_t has at most $4|P| - 8$ edges when $|P| \geq 3$. This is the case of the bound $4N - 8$ for graphs on $N \geq 3$ vertices drawn with at most one crossing per edge [2, 11]; we give a direct proof in the present setting. Let ab and pp' be crossing edges, meeting at o, and let Q be the convex quadrilateral with diagonals ab and pp'. By the argument above, applied to the edge ab, the points p, p' lie in the two boxes for ab, so that, for instance, $|p - a|^2 < (\varpi - \frac{7}{20})^2 + (\varpi - h)^2 \leq \varpi^2$; the last inequality holds because $\varpi^2 - \frac{39}{20}\varpi + \frac{3284}{6400} < 0$ for $\sqrt{2} < \varpi \leq \frac{39}{25}$. In the same way $|p - b|, |p' - a|, |p' - b| < \varpi$, so the four sides of Q are edges of G_t. The argument above also gives $|po|, |p'o| < 1$, and, applied to pp' in place of ab, $|ao|, |bo| < 1$. Every point z of the closed triangle apo satisfies $|z - a| < 1$ or $|z - p| < 1$: on the part of the triangle nearer to a than to p, the convex function $|z - a|$ is maximal at a vertex of that part, and these vertices are a, the midpoint of ap, possibly o, and a point of ao or op equidistant from a and p, all at distance less than 1 from a (because $|ap| < 2$, $|ao| < 1$ and $|po| < 1$); symmetrically for the part nearer to p. Hence the triangle apo contains no point of P other than a and p, and likewise for the other three triangles into which the diagonals cut Q. Next, no edge f crosses the side ap. Otherwise f is not incident to a or p, and the part of f that runs from the crossing point into the triangle apo ends at a point of P outside that triangle, so it leaves the triangle through the segment ao or op. Then f also crosses ab or pp', which contradicts the fact that f is crossed by at most one edge. The same holds for the other three sides of Q.

Let c be the number of crossing pairs of edges, and let G'' be the plane graph obtained from G_t by deleting the $2c$ edges that are crossed. For a crossing pair as above, no edge of G'' meets the interior of Q: it would have an endpoint there, or cross a side of Q, or join two vertices of Q through its interior, and the only such segments are the deleted diagonals. As the sides of Q are edges of G'', the interior of Q is a face of G'' whose boundary walk has length 4. Distinct crossing pairs give distinct faces, because the face determines its four vertices and hence the two diagonals. If $c = 0$ then G_t is plane and has at most $3|P| - 6$ edges. If $c \geq 1$ then G'' contains a cycle, so each of its F faces has a boundary walk of length at least 3, while c of them have length 4. With $e'' = e(G_t) - 2c$, this gives $2e'' \geq 3F + c$ and $2e'' \geq 4c$, and Euler's formula gives $F \geq e'' - |P| + 2$. Hence $e'' \leq 3|P| - 6 - c$, so $2c \leq e'' \leq 3|P| - 6 - c$ and $c \leq |P| - 2$. Therefore

$e(G_t) = e'' + 2c \le 3|P| - 6 + c \le 4|P| - 8$. In all cases with $|P| \ge 3$ the degrees sum to less than $8|P|$, and for $|P| \le 2$ they sum to at most 2. □

For a separation radius ϱ define

$$g_\varrho(t) = \begin{cases} \min(k_\varrho(t), 6), & (t/\varrho)^2 \le 2, \\ \min(k_\varrho(t), 8), & (t/\varrho)^2 > 2 \text{ and } t/\varrho \le \frac{39}{25}, \\ k_\varrho(t), & \text{otherwise.} \end{cases}$$

By Lemmas 31 and 32, $\sum_{z \in P} \nu_P(z,t) \le |P|\, g_\varrho(t)$, where both sides vanish if P is empty. So P has at most $|P|\, g_\varrho(t)/2$ unordered pairs at distance at most $2t$.

Lemma 33 (Close pairs)**.** *Let $0 < \delta_R < \delta$, let $\mathcal{D} \subseteq \mathcal{R} \subseteq X$ be finite sets such that points of $\mathcal{R}$ are more than δ_R apart and points of $\mathcal{D}$ more than δ apart, and put $s = |X \setminus \mathcal{R}|$ and $j = |\mathcal{R} \setminus \mathcal{D}|$. Suppose that $\lfloor (1 + 2\delta/\delta_R)^2 \rfloor = 14$ and that $(u' - v')^2 + u'v'(\frac{2\pi}{9})^2 < \delta_R^2$ for all $u', v' \in \{\delta_R, \delta\}$.*

1. *Every point not in $\mathcal{R}$ is at distance less than δ_R from at most five points of $\mathcal{R}$, and every point not in $\mathcal{D}$ is at distance less than δ from at most five points of $\mathcal{D}$.*
2. *The number of unordered pairs of points of X at distance less than δ_R is at most $5s + \binom{s}{2}$.*
3. *The number of unordered pairs of points of X at distance less than δ is at most $14s + 8j + \binom{s}{2}$.*

Proof. (1) Let (ϱ, P) be $(\delta_R, \mathcal{R})$ or $(\delta, \mathcal{D})$, and let $z \notin P$. If six points of P were at distance less than ϱ from z, two of them, at distances $u', v' < \varrho$ from z, would subtend an angle $\phi \le \pi/3$ at z. Their squared distance would be $u'^2 + v'^2 - 2u'v'\cos\phi \le u'^2 + v'^2 - u'v' \le \max(u', v')^2 < \varrho^2$, contradicting the separation of P.

(2) A pair at distance less than δ_R has a point outside $\mathcal{R}$. At most $\binom{s}{2}$ pairs have both points outside $\mathcal{R}$, and by (1) at most $5s$ have exactly one.

(3) A pair at distance less than δ is not contained in $\mathcal{D}$. At most $\binom{s}{2}$ pairs lie outside $\mathcal{R}$. For $z \notin \mathcal{R}$, the open discs of radius $\delta_R/2$ about the points of $\mathcal{R}$ within distance δ of z are disjoint and lie in the disc of radius $\delta + \delta_R/2$ about z, so there are at most $\lfloor (1 + 2\delta/\delta_R)^2 \rfloor = 14$ such points; this bounds the pairs with exactly one point outside $\mathcal{R}$ by $14s$. The remaining pairs lie in $\mathcal{R}$ and have a point $z \in \mathcal{R} \setminus \mathcal{D}$. The other points of $\mathcal{R}$ within distance δ of z lie in the annulus $\delta_R < |z' - z| \le \delta$. If there were nine of them, two would subtend an angle $\phi \le 2\pi/9$ at z, at distances $u', v' \in [\delta_R, \delta]$, and their

squared distance would be $(u'-v')^2+2u'v'(1-\cos\phi) \le (u'-v')^2+u'v'(\frac{2\pi}{9})^2$. This is a convex function of each of u' and v' separately, so on the square $[\delta_R,\delta]^2$ it is maximal at a corner, where it is less than δ_R^2 by hypothesis. This contradicts the separation of $\mathcal{R}$. So there are at most $8j$ such pairs. $\square$

For $\delta_R = 0.0053$ and $\delta = 0.007348$, the two hypotheses are checked in exact rational arithmetic, using $\pi < \frac{355}{113}$.

7 The forest argument

7.1 Parameters

From now on the wave datum is the datum $W_V = (x_V, \ell_V, r_V)$ of Section 9, and q and $p \ge \bar{p}$ are its blank density and its certified monochromatic bound. We fix

$$a = \tfrac{53}{20000}, \qquad d = \tfrac{1837}{500000}, \qquad w = \tfrac{73}{10000}, \qquad N_u = 16384, \qquad N_s = 8192,$$

and put $\delta_R = 2a = 0.0053$ and $\delta = 2d = 0.007348$.

Unit credit. Put $\Delta = w/N_u$, $t_k = k\Delta$ and $h_0 = 1$, and for $1 \le k \le N_u$ let

$$h_k = \min\big(h_{k-1},\ \Omega(t_k, t_k^2/2)\big).$$

Define the step function $h(t) = h_k$ for $t \in (t_{k-1}, t_k]$, with $h(0) = h_1$. It is nonincreasing.

Short credits. Put $\tau_i = i/N_s$ for $0 \le i \le N_s$. For a radius $\varrho \in \{a, d\}$ and $0 \le i < N_s$ let

$$\omega_\varrho(i) = \Omega\Big(2\varrho\,\frac{1-\tau_i^2}{1+\tau_i^2},\ 2\varrho\,\frac{2\tau_{i+1}}{1+\tau_{i+1}^2}\Big),$$

and set $M_1 = \frac{3}{4}\max_i \omega_a(i)$, $M_2 = \frac{3}{10}\max_i \omega_d(i)$, $c_1(i) = \min(\omega_a(i), M_1)$ and $c_2(i) = \min(\omega_d(i), M_2)$. Put $M_0 = 0$.

For a nonzero vector v let $\varphi(v) \in [0, \pi/2]$ be the angle with $(|v_1|, |v_2|) = |v|(\cos\varphi, \sin\varphi)$, and let $i(v)$ be the index with $\tan(\varphi(v)/2) \in [\tau_i, \tau_{i+1})$ (with $i = N_s - 1$ if $\varphi = \pi/2$).

Lemma 34. *Let z be uniform on $\mathbb{R}^2/\Lambda$.*

1. *If $0 < |v| \le 2\varrho$ then $\Pr(z, z+v$ both blank$) \ge \omega_\varrho(i(v))$.*
2. *If $v = (\sin t, \cos t)$ with $|t| \le w$ then $\Pr(z, z+v$ both blank$) \ge h(|t|)$.*

Proof. (1) With $\varphi = \varphi(v) \in [2\arctan\tau_i, 2\arctan\tau_{i+1}]$ we have $|v_1| \le 2\varrho\cos\varphi \le 2\varrho\frac{1-\tau_i^2}{1+\tau_i^2}$ and $|v_2| \le 2\varrho\sin\varphi \le 2\varrho\frac{2\tau_{i+1}}{1+\tau_{i+1}^2}$. Apply Lemma 19 and Lemma 17(3). (2) If $|t| \in (t_{k-1}, t_k]$, or $t = 0$ and $k = 1$, then $v - (0,1) = (\sin t, \cos t - 1)$ has $|\sin t| \le t_k$ and $1 - \cos t \le t_k^2/2$. By Lemma 17(3) and Lemma 19 the probability is at least $\Omega(t_k, t_k^2/2) \ge h_k$. □

7.2 The candidate graph

Let X be a finite set of n points, with m unit pairs, all of which we regard as edges. Choose a maximal subset $\mathcal{R} \subseteq X$ with pairwise distances greater than $2a$, and a maximal subset $\mathcal{D} \subseteq \mathcal{R}$ with pairwise distances greater than $2d$. Put $s = |X \setminus \mathcal{R}|$ and $j = |\mathcal{R} \setminus \mathcal{D}|$. By maximality each $x \in X \setminus \mathcal{R}$ has a *parent* $\pi(x) \in \mathcal{R}$ with $|x - \pi(x)| \le 2a$, and each $x \in \mathcal{R} \setminus \mathcal{D}$ has a parent $\pi(x) \in \mathcal{D}$ with $|x - \pi(x)| \le 2d$. Iterating π at most twice from x reaches a *root* $\mathrm{rt}(x) \in \mathcal{D}$; let $\mathrm{dp}(x) \in \{0,1,2\}$ be the number of steps. Then

$$|x - \mathrm{rt}(x)| \le 2d \ \ (x \in \mathcal{R}), \qquad |x - \mathrm{rt}(x)| \le 2a + 2d \ \ (x \in X). \tag{5}$$

The sets $\mathcal{R}, \mathcal{D}$ and the parents are fixed once and for all. Define the *layer* $\lambda(x)$ to be 1 if $x \notin \mathcal{R}$, 2 if $x \in \mathcal{R} \setminus \mathcal{D}$, and 0 if $x \in \mathcal{D}$.

Consider a random placement with rotation ϑ and write $\hat{x} = \rho_\vartheta x + \tau$. Let $\mathcal{E} \in \{\mathcal{R}, \mathcal{D}\}$ be the set of *eligible tails*; the two choices are called *mode* $\mathcal{R}$ and *mode* $\mathcal{D}$. For each ϑ, the *candidate edges* are directed pairs (x, y), from a *tail* x to a *head* y, of three kinds:

- the *short edge* $(\pi(y), y)$, for each $y \notin \mathcal{D}$, with credit $c_{\lambda(y)}\big(i(\hat{y} - \widehat{\pi(y)})\big)$;
- each *unit edge* (x, y) with $x \in \mathcal{E}$, $|x - y| = 1$ and $\hat{y} - \hat{x} = (\sin t, \cos t)$ for some $|t| \le w$, with credit $h(|t|)$; we write $t = t_{xy}$;
- the *centre edges* defined in Section 8, whose heads lie in $\mathcal{D}$.

Short and unit edges are *ordinary*. Candidate edges depend on ϑ but not on τ. Assign to x the pair $\Phi(x) = \big(\text{height of } \rho_\vartheta\,\mathrm{rt}(x),\ \mathrm{dp}(x)\big)$, ordered lexicographically, using the rotated but untranslated root; it depends on ϑ but not on τ. Since translation by τ raises all heights by the same amount, the height of $\widehat{\mathrm{rt}(y)}$ exceeds that of $\widehat{\mathrm{rt}(x)}$ exactly when the first coordinate of $\Phi(y)$ exceeds that of $\Phi(x)$.

Lemma 35. *For every ϑ, Φ strictly increases from tail to head along every candidate edge. Consequently, every set of candidate edges in which each vertex is the head of at most one edge is a forest.*

Proof. A short edge $(\pi(y), y)$ preserves the root and increases the depth by one. For a unit edge (x, y), the tail lies in $\mathcal{E} \subseteq \mathcal{R}$ and the height of $\hat{y}$ exceeds that of $\hat{x}$ by $\cos t \geq \cos w$, so by (5) the height of $\widehat{\mathrm{rt}(y)}$ exceeds that of $\widehat{\mathrm{rt}(x)}$ by at least

$$\cos w - 2d - (2a + 2d) = \cos w - 4d - 2a > 0.9999 - 0.0200 > 0.$$

For centre edges see Lemma 47(3). So there is no directed cycle. A cycle of the underlying undirected graph, including a pair of opposite edges, has as many edges as vertices; if each vertex is the head of at most one edge, every vertex of the cycle receives its incoming edge from the cycle, giving a directed cycle. Hence the set is a forest. □

The centre edges depend on the choice of one of two profiles, made in Section 8; we fix a profile. We also use a third variant, with no ordinary edges at all, in which the centre edges are the only candidates. For each placement let F consist of one candidate edge entering y whose two ends are blank, for each vertex y that is the head of such an edge, chosen by a fixed rule. By Lemma 35, F is a forest, and it depends on the placement. This is the forest to which we apply Lemma 20.

Lemma 36 (Two tails in $\mathcal{D}$)**.** *Let $x \neq x'$ be points of $\mathcal{D}$ such that (x, y) and (x', y) are unit candidate edges for the same ϑ. Then, for almost every τ, the points $\hat{x}$ and $\hat{x}'$ are not both blank.*

Proof. Write $\hat{y} - \hat{x} = (\sin t, \cos t)$ and $\hat{y} - \hat{x}' = (\sin t', \cos t')$ with $|t|, |t'| \leq w$. Then $\hat{x}' - \hat{x}$ has horizontal component of modulus $|\sin t - \sin t'| \leq 2w$ and vertical component of modulus $|\cos t - \cos t'| \leq 1 - \cos w \leq w^2/2$. Up to a null set of τ, both points are blank only if each lies in a translate $\zeta + O$, $\zeta \in \Gamma$, by Lemma 5(3). These translates lie in vertical columns centred on the lines $u \in 2x\mathbb{Z}$, each of width at most $W = 2\ell_V(0) = 0.007347530728$. Since $2w + W < 2x$, both points lie in the same column, so $|\hat{x} - \hat{x}'|^2 \leq W^2 + w^4/4 < \delta^2$. This contradicts $|x - x'| > 2d = \delta$. Both inequalities are checked in exact arithmetic. □

7.3 Accounting at a head

Lemma 37. *Let $0 \leq c \leq M$, let $h_1, \ldots, h_r \geq 0$, and let $\varepsilon \geq 0$. Then*

$$\max\Big(c,\ \sum_i h_i - \varepsilon\Big) \geq c + \sum_i [h_i - M]_+ - \varepsilon.$$

Proof. If $\sum_i [h_i - M]_+ \le \varepsilon$ the right side is at most c. Otherwise some h_i exceeds M, so $\sum_i \min(h_i, M) \ge M \ge c$, and $\sum_i h_i - \varepsilon = \sum_i \min(h_i, M) + \sum_i [h_i - M]_+ - \varepsilon$. □

Lemma 38 (Accounting at a head)**.** *Fix ϑ and a vertex y. Let c be the credit of the short edge entering y ($c = 0$ if $y \in \mathcal{D}$), let $h_1, \dots, h_r$ be the credits of the unit candidate edges entering y, and let P_o be the probabilities, over τ, of the events $\mathcal{E}_o$ attached in Section 8 to the centre candidate edges o entering y. (In the variant without ordinary edges, take $c = 0$ and $r = 0$.) Then*

$$\Pr_{\tau}\left(F \text{ has an edge entering } y \;\middle|\; \vartheta\right) \ge c + \sum_i \left[h_i - M_{\lambda(y)}\right]_+ - {\sum_{\{i,i'\}}}' \min(h_i, h_{i'}) + \sum_o P_o,$$

where the primed sum is over unordered pairs of unit candidate edges entering y whose tails are not both in $\mathcal{D}$.

Proof. Let $\mathcal{E}_0$ be the event that the short edge entering y has both ends blank, and $\mathcal{E}_i$ the same event for the i-th unit edge. By Lemma 34 and the fact that, for fixed ϑ, $\hat{x}$ is uniform on $\mathbb{R}^2/\Lambda$ for each x, we have $\Pr(\mathcal{E}_0) \ge c$ and $\Pr(\mathcal{E}_i) \ge h_i$. Haar measure on $\mathbb{R}^2/\Lambda$ has no atoms, so there are events $\mathcal{E}_0' \subseteq \mathcal{E}_0$ and $\mathcal{E}_i' \subseteq \mathcal{E}_i$ with $\Pr(\mathcal{E}_0') = c$ and $\Pr(\mathcal{E}_i') = h_i$. By Lemma 36, $\Pr(\mathcal{E}_i' \cap \mathcal{E}_{i'}') = 0$ when both tails lie in $\mathcal{D}$, and in any case $\Pr(\mathcal{E}_i' \cap \mathcal{E}_{i'}') \le \min(h_i, h_{i'})$. Let $\mathcal{A} = \mathcal{E}_0 \cup \bigcup_i \mathcal{E}_i$. By Bonferroni's inequality and Lemma 37 with $M = M_{\lambda(y)} \ge c$,

$$\begin{aligned}\Pr(\mathcal{A}) &\ge \max\Big(c,\ \sum_i h_i - {\sum}' \min(h_i, h_{i'})\Big) \\ &\ge c + \sum_i [h_i - M]_+ - {\sum}' \min(h_i, h_{i'}).\end{aligned}$$

By Lemma 47(2) at most one centre candidate edge enters y, and by Lemma 48, up to a null set its event $\mathcal{E}_o$ is disjoint from $\mathcal{A}$ and implies that both ends of o are blank. On $\mathcal{A} \cup \mathcal{E}_o$ some candidate edge entering y has both ends blank, so F has an edge entering y. □

Every short credit entering y is at most $M_{\lambda(y)}$. Summing Lemma 38 over y and averaging over ϑ,

$$\begin{aligned}\mathbb{E}\,|F| \;\ge\; \sum_{y \notin \mathcal{D}} \mathbb{E}\,\operatorname{cred}(\pi(y), y) &+ \mathbb{E}\sum_{(x,y)} \left[h(|t_{xy}|) - M_{\lambda(y)}\right]_+ \\ &- \mathbb{E}\sum_y {\sum_{\{x,x'\}}}' \min\left(h(|t_{xy}|), h(|t_{x'y}|)\right) + \mathcal{C}_{\mathrm{ctr}}, \quad (6)\end{aligned}$$

where the second sum is over the unit candidate edges for the given ϑ, the third over unordered pairs of unit candidate edges entering the same head whose tails are not both in $\mathcal{D}$, and $\mathcal{C}_{\mathrm{ctr}}$ is the total probability of the centre events, bounded below in Proposition 52. In mode $\mathcal{D}$ the third sum is empty.

7.4 Expected credits

Define

$$B_0 = \frac{2}{\pi}\int_0^w h(t)\,\mathrm{d}t = \frac{2\Delta}{\pi}\sum_{k=1}^{N_u} h_k, \qquad \rho(M) = \frac{\sum_k [h_k - M]_+}{\sum_k h_k},$$

and $\rho(v) = \rho(M_{\lambda(v)})$ for $v \in X$, so that $\rho(v) = 1$ for $v \in \mathcal{D}$.

Lemma 39 (Short credits). *For y in layer $\lambda \in \{1,2\}$, $\mathbb{E}_\vartheta \operatorname{cred}(\pi(y), y) = C^0_\lambda$, where*

$$C^0_\lambda = \frac{2}{\pi}\sum_{i=0}^{N_s-1} c_\lambda(i)\big(2\arctan\tau_{i+1} - 2\arctan\tau_i\big) \;\geq\; \frac{2}{\pi_{\mathrm{hi}}}\sum_{i=0}^{N_s-1} c_\lambda(i)\,\frac{2(\tau_{i+1}-\tau_i)}{1+\tau_{i+1}^2}.$$

Proof. Under a uniform rotation the folded angle $\varphi(\hat{y} - \widehat{\pi(y)})$ is uniform on $[0, \pi/2]$. The inequality uses $\arctan v - \arctan u \geq (v-u)/(1+v^2)$ for $0 \leq u \leq v$. □

Lemma 40 (Unit credits). *For $v \in X$ let $w_{\mathcal{E}}(v) = \mathbf{1}[v \notin \mathcal{E}] + 1 - \rho(v)$. Then*

$$\mathbb{E}_\vartheta \sum_{(x,y)} \big[h(|t_{xy}|) - M_{\lambda(y)}\big]_+ \;\geq\; B_0 \max\Big(0,\; m - \sum_{v\in X} \tfrac{1}{2} w_{\mathcal{E}}(v)\, d_v\Big).$$

Proof. Fix a unit pair $\{x, y\}$. Under a uniform rotation the signed angle t of $\hat{y} - \hat{x}$ from the upward vertical is uniform on the circle, and (x, y) is a candidate exactly when $x \in \mathcal{E}$ and $|t| \leq w$. So the expected contribution of the orientation (x, y) is

$$\mathbf{1}[x \in \mathcal{E}]\,\frac{1}{2\pi}\int_{-w}^{w} \big[h(|t|) - M_{\lambda(y)}\big]_+ \,\mathrm{d}t$$
$$= \mathbf{1}[x \in \mathcal{E}]\,\frac{\Delta}{\pi}\sum_k \big[h_k - M_{\lambda(y)}\big]_+ = \frac{B_0}{2}\,\mathbf{1}[x \in \mathcal{E}]\,\rho(y).$$

Since $0 \leq \rho \leq 1$ we have $\mathbf{1}[x \in \mathcal{E}]\rho(y) \geq 1 - \mathbf{1}[x \notin \mathcal{E}] - (1 - \rho(y))$. Adding both orientations, the pair contributes at least $\frac{B_0}{2}\big(2 - w_{\mathcal{E}}(x) - w_{\mathcal{E}}(y)\big)$, and also at least 0. Summing over the m pairs and collecting terms by vertex gives the claim. □

Write $\rho_\lambda = \rho(M_\lambda)$ for $\lambda = 1, 2$. In mode $\mathcal{R}$ the weights $\frac{1}{2} w_{\mathcal{E}}(v)$ are $\alpha = 1 - \rho_1/2$ on $X \setminus \mathcal{R}$, $\beta_{\mathcal{R}} = (1 - \rho_2)/2$ on $\mathcal{R} \setminus \mathcal{D}$, and 0 on $\mathcal{D}$. In mode $\mathcal{D}$ they are α, $\beta_{\mathcal{D}} = 1 - \rho_2/2$ and 0.

Lemma 41 (Collisions). *Let $x \neq x'$ be eligible tails with a common unit neighbour y. Then*

$$\mathbb{E}_\vartheta \Big[\min\big(h(|t_{xy}|), h(|t_{x'y}|)\big)\, \mathbf{1}[(x,y) \text{ and } (x',y) \text{ are both candidates}] \Big] \leq \frac{\Delta}{\pi} \sum_{k:\, 2t_k \geq |x - x'|} h_k.$$

Consequently the expectation of the third sum in (6) is at most $\frac{2\Delta}{\pi} \sum_k h_k \mathcal{N}'(2t_k)$, where $\mathcal{N}'(\varrho)$ is the number of unordered pairs of points of $\mathcal{R}$, not both in $\mathcal{D}$, at distance at most ϱ.

Proof. Let $\phi \in (0, \pi]$ be the angle at y between $x - y$ and $x' - y$, so $|x - x'| = 2\sin(\phi/2)$. If t is the signed angle of $\hat{y} - \hat{x}$ from the vertical, then that of $\hat{y} - \hat{x}'$ is $t + \phi$ or $t - \phi$; by symmetry take $t + \phi$. Put $\tilde{h}(t) = h(|t|)\mathbf{1}[|t| \leq w]$, which is even and nonincreasing in $|t|$. The expectation equals $\frac{1}{2\pi} \int \tilde{h}\big(\max(|t|, |t + \phi|)\big)\, \mathrm{d}t$. For $u \geq 0$ the set where $\max(|t|, |t + \phi|) \leq u$ is an interval of length $[2u - \phi]_+$, which is empty for $u < \phi/2$. If $\phi/2 \geq w$, the integrand vanishes except on a null set and the expectation is 0. Otherwise the integral equals $2\int_{\phi/2}^{w} h(u)\, \mathrm{d}u$, and the expectation is $\frac{1}{\pi} \int_{\phi/2}^{w} h(u)\, \mathrm{d}u \leq \frac{\Delta}{\pi} \sum_{k:\, t_k \geq \phi/2} h_k$. In both cases the claimed bound holds, and $t_k \geq \phi/2$ implies $2t_k \geq 2\sin(\phi/2) = |x - x'|$.

In mode $\mathcal{R}$ the tails of the pairs in the third sum lie in $\mathcal{R}$ and are not both in $\mathcal{D}$; in mode $\mathcal{D}$ there are none. Two distinct points have at most two common unit neighbours, which gives the factor 2. □

Lemma 42 (Collision constants). *Let k_ϱ and g_ϱ be as in Section 6, and define*

$$L^0 = \frac{\Delta}{\pi} \sum_k g_a(t_k) h_k, \qquad T^0 = \frac{\Delta}{\pi} \sum_{k:\, t_k > d} g_a(t_k) h_k,$$
$$U^0 = \frac{2\Delta}{\pi} \sum_{k:\, t_k \leq d} k_a(t_k) h_k, \qquad J^0 = \frac{2\Delta}{\pi} \sum_k k_a(t_k) h_k.$$

Then $\frac{2\Delta}{\pi} \sum_k h_k \mathcal{N}'(2t_k) \leq \min\big((n - s)L^0,\ (n - s)T^0 + jU^0,\ jJ^0\big)$.

Proof. The set $\mathcal{R}$ is $2a$-separated with $|\mathcal{R}| = n - s$. By Section 6, $\mathcal{N}'(2t)$ is at most the number of pairs of $\mathcal{R}$ at distance at most $2t$, which is at most

$(n-s)g_a(t)/2$; this gives the first bound. Every pair counted by $\mathcal{N}'(2t)$ has a point in $\mathcal{R}\setminus\mathcal{D}$, and by Lemma 31 each point of $\mathcal{R}\setminus\mathcal{D}$ is within $2t$ of at most $k_a(t)$ other points of $\mathcal{R}$. So $\mathcal{N}'(2t)\le j\,k_a(t)$, which gives the third bound. For the second, use $\mathcal{N}'(2t_k)\le j\,k_a(t_k)$ when $t_k\le d$ and $\mathcal{N}'(2t_k)\le (n-s)g_a(t_k)/2$ when $t_k>d$. □

7.5 The main inequality

For weights $\alpha,\beta\ge 0$ define

$$\Lambda_{\alpha,\beta}(m,s,j)=\frac{2m}{n}(\alpha s+\beta j)+\sqrt{2V(m)\Big(\alpha^2 s+\beta^2 j-\frac{(\alpha s+\beta j)^2}{n}\Big)}.$$

Lemma 43. *Under the hypotheses of Lemma 29 with $k=2090$, let $z_v\ge 0$ be weights taking the value α on a set of s vertices, β on a disjoint set of j vertices, and 0 otherwise. Then $\sum_v z_v d_v\le\Lambda_{\alpha,\beta}(m,s,j)$. In particular, the sum of the degrees of any h vertices is at most $\Lambda_{1,0}(m,h,0)=\frac{2mh}{n}+\sqrt{2V(m)h(1-h/n)}$.*

Proof. Write $\bar d=2m/n$ and $\bar z=\frac{1}{n}\sum z_v$. Then $\sum z_v d_v=\bar d\sum z_v+\sum(z_v-\bar z)(d_v-\bar d)$. Apply the Cauchy–Schwarz inequality, the identity $\sum(z_v-\bar z)^2=\sum z_v^2-(\sum z_v)^2/n$, and Lemma 29. □

Proposition 44 (Main inequality)**.** *Let X be a set of $n\ge 2090$ points whose unit-distance graph has m edges, has no proper 6-colouring, and has every non-isolated vertex of degree at least 6. Let s,j be as in Section 7.2. Let $0\le B\le B_0$, $C\le C_1^0$ and $E\le C_2^0$, let $L\ge L^0$, $T\ge T^0$, $U\ge U^0$, $J\ge J^0$, and put $p'=p-\beta_V/2$. Then*

$$1\le\Psi(m,s,j):=\min\big\{nq+mp',\ \Psi_{\mathcal{R}}(m,s,j),\ \Psi_{\mathcal{D}}(m,s,j)\big\}-\mathcal{C}(m,s,j),$$

where $\mathcal{C}$ is the centre credit of Proposition 52,

$$\begin{aligned}\Psi_{\mathcal{R}}&=\Psi_0+B\,\Lambda_{\alpha,\beta_{\mathcal{R}}}(m,s,j)+\min\big((n-s)L,\ (n-s)T+jU,\ jJ\big),\\ \Psi_{\mathcal{D}}&=\Psi_0+B\,\Lambda_{\alpha,\beta_{\mathcal{D}}}(m,s,j),\\ \Psi_0&=nq+m(p'-B)-sC-jE.\end{aligned}$$

Proof. Fix a profile. Apply (3) to a random placement in the wave colouring of W_V and the forest F of Section 7.2, built with no ordinary edges, or in mode $\mathcal{R}$, or in mode $\mathcal{D}$. In each case $\mathbb{E}Y=nq-\mathbb{E}|F|$ and $\mathbb{E}X\le mp$, and Proposition 25 gives $\frac14\sum_v\mathbb{E}(C_v-L_v)\ge\frac14\beta_V\cdot 2m$. So $1\le nq+mp'-\mathbb{E}|F|$. With no ordinary edges, $\mathbb{E}|F|\ge\mathcal{C}_{\mathrm{ctr}}$ by Lemma 38. In the two modes, bound $\mathbb{E}|F|$ by (6), Lemma 39, Lemma 40, Lemma 41, Lemma 42 and Lemma 43;

in Lemma 40 we use $B_0 \max(0, m-\Lambda') \geq B(m-\Lambda')$ for every real Λ', since $0 \leq B \leq B_0$. Finally, choosing the profile that gives the larger value, $\mathcal{C}_{\mathrm{ctr}} \geq \mathcal{C}(m,s,j)$ by Proposition 52 and the definition following it. □

The ratios ρ_1, ρ_2 enter only through $\alpha, \beta_{\mathcal{R}}, \beta_{\mathcal{D}}$, which decrease as ρ_1, ρ_2 increase, so lower bounds for ρ_1, ρ_2 suffice.

8 Edges between circle centres

Throughout this section X is as in Proposition 44, with $\mathcal{R}, \mathcal{D}, s, j$ as in Section 7.2. Put $h = |X \setminus \mathcal{D}| = s + j$, and let $L_{\mathcal{R}} = \sum_{x \notin \mathcal{R}} d_x$ and $L_{\mathcal{D}} = \sum_{x \notin \mathcal{D}} d_x$ be the degree sums outside $\mathcal{R}$ and $\mathcal{D}$. Recall $\delta_R = 2a$ and $\delta = 2d$.

8.1 Doubled arcs

As in the proof of Lemma 29, for each non-isolated $v \in X$ the $d_v \geq 3$ neighbours of v, which we call the *marked points* of the unit circle S_v about v, divide S_v into d_v closed *arcs*, each joining two consecutive marked points and containing no marked point in its interior. There are $2m$ arcs. Since every unit pair in X is an edge, a point of X on S_v is a marked point. The *angle* z of an arc is its length, and its *chord* is the distance between its endpoints; always $z \geq$ chord.

Two points $p \neq q$ at distance $\ell < 2$ have exactly two common unit neighbours in the plane, and a circle with at least three marked points has at most one arc joining p and q. So at most two arcs join p and q. If two do, we call $\{p, q\}$ a *doubled pair*; the two arcs lie on S_u and S_v, where $u \neq v$ are the common unit neighbours of p and q, which we call the *centres* of the pair, and ℓ is its *chord*. Let t be the number of doubled pairs.

Lemma 45. *Let $k = 1317$, let P_2 and P_4 be as in Lemma 27, and put*

$$A^*(m) = \left\lfloor \frac{\bigl(n(n-1) + m - 2m^2/n\bigr)P_4 + \frac{266}{9}(k-2)}{6P_2} \right\rfloor.$$

Then $t \geq 2m - A^*(m)$.

Proof. Keep one arc of each doubled pair and all other arcs. This is a drawing of a simple graph G' on X with $A = 2m - t$ edges, in which no arc passes through a vertex. As in the proof of Lemma 29, any two of its arcs have finitely many common points, and it has at most $n(n-1) - \sum_u \binom{d_u}{2}$ crossings. Since $\sum_u d_u^2 \geq (2m)^2/n$, this is at most $n(n-1) + m - 2m^2/n$, so $\mathrm{cr}(G')$ is

at most this number. By Lemma 27 with $k = 1317 \le n$, $\mathrm{cr}(G') \ge (6AP_2 - \frac{266}{9}(k-2))/P_4$. Hence $A \le A^*(m)$, since A is an integer. □

Lemma 46 (Rhombus). *Let $\{p, q\}$ be a doubled pair with centres u, v and chord ℓ, and put $M = \frac{p+q}{2}$, $e = \frac{q-p}{\ell}$ and $\alpha_\ell = \arcsin(\ell/2)$. Then $M = \frac{u+v}{2}$, $|u - v| = r_\ell := \sqrt{4 - \ell^2}$ and $(u - v) \cdot e = 0$. The arc of S_u joining p and q is either the* minor arc, *consisting of the points $u + e_\psi$ with e_ψ at angle at most α_ℓ from $v - u$, or it has angle at least π. A point $P \in S_u$ with $(P - u) \cdot (v - u) \ge 0$ lies in the interior of the minor arc if and only if $|(P - M) \cdot e| < \ell/2$.*

Proof. The points u, p, v, q form a rhombus with unit sides, whose diagonals uv and pq bisect each other at right angles. So $M = \frac{u+v}{2}$, $|u - v|^2 = 4 - \ell^2$, and $p - u$ and $q - u$ make angles $\pm\alpha_\ell$ with $f = (v - u)/r_\ell$. The two arcs of S_u between p and q have angles $2\alpha_\ell < \pi$ and $2\pi - 2\alpha_\ell > \pi$. Write $P - u = \cos\psi'\, f + \sin\psi'\, e$ with $|\psi'| \le \pi/2$, which is possible exactly when $(P - u) \cdot f \ge 0$. Since $(u - M) \cdot e = 0$, we get $(P - M) \cdot e = \sin\psi'$, and P is interior to the minor arc if and only if $|\psi'| < \alpha_\ell$, that is $|\sin\psi'| < \ell/2$. □

8.2 Centre candidate edges

We use two *profiles* (I, I_R, J), the *coarse* profile $(23,\ 7.9,\ 98) \cdot 10^{-9}$ and the *fine* profile $(25.9,\ 9.3,\ 110) \cdot 10^{-9}$. For each profile we fix a list of chord intervals covering $[\delta_R, 0.245]$, and assign each chord ℓ in this range to the first interval in the list that contains it.

- *Small intervals*, with parameters $(\sigma, \ell_+, N_\theta, N_y)$ where σ is at most the left end and ℓ_+ at least the right end of the interval. For the coarse profile: $[\delta_R, \delta)$ with $(\delta_R, 0.02, 32, 48)$ and $[\delta, 0.02]$ with $(\delta, 0.02, 32, 48)$. For the fine profile: $[\delta_R, 0.006]$ with $(\delta_R, 0.006, 100, 180)$, $(0.006, \delta)$ with $(0.006, 0.02, 100, 180)$, $[\delta, 0.008]$ with $(\delta, 0.008, 100, 180)$ and $(0.008, 0.02]$ with $(0.008, 0.02, 100, 180)$. Put $\eta = \min(0.00365, \sigma/2)$.
- *Medium intervals* $(l, l']$, where $l < l'$ are consecutive points of $\{0.02, 0.04, 0.08, 0.12, 0.16, 0.18\} \cup \{0.190, 0.191, \dots, 0.245\}$; there are 61 of them. Put $\eta = \min(0.0036,\ l/2 - w)$, and $N_\theta = 48$ for the coarse and $N_\theta = 256$ for the fine profile.

Fix ϑ. An *orientation* of a doubled pair is an ordered pair (v, u) of its centres; u is its *head*. Let $\theta \in (-\pi, \pi]$ be the signed angle from $(0, 1)$ to $\hat{u} - \hat{v}$. The orientation gives the *centre candidate edge* (v, u) if $u \in \mathcal{D}$, the chord ℓ lies in $[\delta_R, 0.245]$, the arc of S_u joining p and q is the minor arc, and $|\theta| \le \eta$ for the interval of ℓ.

Lemma 47. *Let (v,u) be a centre candidate edge for ϑ, from a doubled pair $\{p,q\}$ with chord ℓ.*

1. *The point $\hat{u}-(0,1)$ lies in the interior of the image of the arc of S_u joining p and q.*
2. *No other centre candidate edge enters u.*
3. *$\Phi(u)>\Phi(v)$.*
4. *If ℓ lies in a medium interval, no unit candidate edge enters u.*

Proof. (1) The direction of $\hat{v}-\hat{u}$ makes the angle $|\theta|$ with $-(0,1)$, and $|\theta|\le\eta\le\ell/2<\alpha_\ell$: for small intervals $\eta\le\sigma/2\le\ell/2$, and for medium intervals $\eta<l/2<\ell/2$. Apply Lemma 46 after rotation.

(2) A centre candidate edge (v',u) determines the arc of S_u joining the endpoints of its pair, and conversely this arc determines the pair and v'. By (1), $\hat{u}-(0,1)$ is interior to that arc. Distinct arcs of S_u have disjoint interiors.

(3) Since $u\in\mathcal{D}$, $\mathrm{rt}(u)=u$. The height of $\hat{u}$ exceeds that of $\hat{v}$ by $r_\ell\cos\theta\ge\sqrt{4-0.245^2}\,\cos(0.00365)>1.9\cdot0.9999>2a+2d$. By (5) the height of $\widehat{\mathrm{rt}(v)}$ is less than that of $\hat{u}$.

(4) Let (x,u) be a unit candidate edge. The direction of $\hat{x}-\hat{u}$ is within w of $-(0,1)$, hence within $w+\eta\le l/2<\alpha_\ell$ of the direction of $\hat{v}-\hat{u}$. By Lemma 46, $\hat{x}$ is interior to the image of the minor arc of S_u, which is the arc joining p and q. But x is a neighbour of u, hence a marked point, and arcs contain no marked point in their interior. □

Let $W(y)=\ell_V(|y|)$ for $|y|\le y_0$ and $W(y)=0$ otherwise, so that $O=\{(u,y):|y|\le y_0,\ |u|\le W(y)\}$ and W is nonincreasing in $|y|$. To each centre candidate edge $o=(v,u)$ we attach an event $\mathcal{E}_o$, a set of translations τ:

- *Medium chord:* $\hat{v}\in\zeta+O$ and $\hat{u}\in\zeta+(0,2)+O$ for some $\zeta\in\Gamma$.
- *Small chord:* with $(\sigma,\ell_+,N_\theta,N_y)$ the parameters of the interval of ℓ, put $\Delta_\theta=\eta/N_\theta$, $\Delta_y=0.009/N_y$, $\theta_i=i\Delta_\theta$ and $y_k=k\Delta_y$; let $a_i\ge1-\sqrt{1-\ell_+^2/4+\theta_i^2/2}$ be a rational upper bound and $b_i=a_i+w^2/2$; and put

  $$w_{i,k}=\Big[\min\Big(W(y_k+\Delta_y+a_i)-\theta_i,\ \frac{\sigma}{2}-W\big(\max(0,y_k-b_i)\big)-b_i\theta_i\Big)\Big]_+.$$

  If $|\theta|\in((i-1)\Delta_\theta,i\Delta_\theta]$ (or $\theta=0$ and $i=1$), let $\mathcal{R}_\theta$ be the union over $0\le k<N_y$ of the sets $\{(M_1,M_2):|M_1|<w_{i,k},\ y_k\le|M_2|\le y_k+\Delta_y\}$. The event is $\hat{M}\in\zeta+\mathcal{R}_\theta$ for some $\zeta\in\Gamma$, where M is the midpoint of the pair.

The blank set is invariant under translation by Γ, since Γ is generated by Λ and $(0, H)$, which permute the colour classes (Lemma 6(1)). The program checks that

$$2 \max W + \eta + w < 2x \qquad \text{and} \qquad 0.009 + b_{N_\theta} + y_0 < \tfrac{1}{2} \tag{7}$$

for every small interval.

Lemma 48. *Let $o = (v, u)$ be a centre candidate edge for ϑ. For almost every $\tau \in \mathcal{E}_o$, the points $\hat{u}$ and $\hat{v}$ are blank and no unit candidate edge entering u has a blank tail.*

Proof. By Lemma 5(3), every translate $\zeta + O$ with $\zeta \in \Gamma$ consists of blank points, and $(0, 1) \in \Gamma$. For a medium chord this gives the first claim, and Lemma 47(4) the second.

For a small chord, by Γ-invariance we may take $\zeta = 0$. Let $|\theta| \in ((i-1)\Delta_\theta, i\Delta_\theta]$ and $\hat{M} = (M_1, M_2)$ with $|M_1| < w_{i,k}$ and $y_k \le |M_2| \le y_k + \Delta_y$. With $f = (-\sin\theta, \cos\theta)$ we have $\hat{u} = \hat{M} + \frac{r_\ell}{2} f$ and $\hat{v} = \hat{M} - \frac{r_\ell}{2} f$. Since $\frac{r_\ell}{2} = \sqrt{1 - \ell^2/4} \ge \sqrt{1 - \ell_+^2/4}$,

$$0 \le 1 - \tfrac{r_\ell}{2} \cos\theta \le \left(1 - \tfrac{r_\ell}{2}\right) + (1 - \cos\theta) \le a_i.$$

So $\hat{u} - (0, 1)$ has horizontal coordinate of modulus at most $|M_1| + \theta_i < W(y_k + \Delta_y + a_i)$ and vertical coordinate of modulus at most $y_k + \Delta_y + a_i$. As W is nonincreasing in $|y|$, $\hat{u} - (0, 1) \in O$, and $\hat{u}$ is blank. In the same way $\hat{v} + (0, 1) \in O$.

Let (x, u) be a unit candidate edge, so $\hat{x} = \hat{u} - (\sin t', \cos t')$ with $|t'| \le w$. Then $\hat{x}_2 - M_2 = \frac{r_\ell}{2} \cos\theta - \cos t' \in [-a_i, w^2/2]$, so $|\hat{x}_2 - M_2| \le b_i$, and $|\hat{x}_1| \le |M_1| + \theta_i + w$. By (7) the only translate $\zeta' + O$, $\zeta' \in \Gamma$, that can contain $\hat{x}$ is O itself: those with ζ' in another column are more than $2x - 2\max W - \eta - w > 0$ away horizontally, and those in the same column are centred at $(0, \pm\frac{1}{2}), (0, \pm 1), \ldots$ and are more than $\frac{1}{2} - 0.009 - b_i - y_0 > 0$ away vertically. So, outside a null set of τ, if $\hat{x}$ is blank then $\hat{x} \in O$ and $|\hat{x}_1| \le W(\hat{x}_2) \le W(\max(0, y_k - b_i))$. Let $\hat{e} = (\cos\theta, \sin\theta)$, the image of $\pm e$. Then

$$\begin{aligned} |(\hat{x} - \hat{M}) \cdot \hat{e}| &\le |\hat{x}_1 - M_1| + |\hat{x}_2 - M_2|\,|\sin\theta| \\ &\le W(\max(0, y_k - b_i)) + |M_1| + b_i\theta_i < \frac{\sigma}{2} \le \frac{\ell}{2}. \end{aligned}$$

Also $(\hat{x} - \hat{u}) \cdot (\hat{v} - \hat{u}) > 0$, since the directions differ by at most $w + |\theta| < \pi/2$. By Lemma 46, $\hat{x}$ is interior to the image of the minor arc of S_u, which is the arc joining p and q. As in Lemma 47(4) this is impossible. □

Lemma 49 (Centre probabilities)**.** *Let (v,u) be an orientation of a doubled pair with chord $\ell \geq \delta_R$ and $u \in \mathcal{D}$, and let z_u be the angle of the arc of S_u joining its endpoints. For each profile, the probability over the placement that (v,u) is a centre candidate edge and $\tau \in \mathcal{E}_{(v,u)}$ is at least $\frac{1}{2}\big(I - J(z_u - \delta)\big)$ if $\ell \geq \delta$, and at least $\frac{1}{2}\big(I_R - J(z_u - \delta_R)\big)$ if $\ell < \delta$.*

Proof. If $\ell > 0.245$ then $z_u \geq \ell > 0.245$, and if the arc is not the minor arc then $z_u \geq \pi > 0.245$ by Lemma 46. In both cases the bound is negative because $I - J(0.245 - \delta) < 0$ and $I_R - J(0.245 - \delta_R) < 0$ for both profiles. Otherwise, θ is uniform on $(-\pi, \pi]$, and for fixed ϑ the points $\hat{v}$ and $\hat{M}$ are uniform on $\mathbb{R}^2/\Lambda$.

Medium chord in $(l, l']$. For fixed θ, the events $\{\hat{v} \in \zeta + O,\ \hat{u} \in \zeta + (0,2) + O\}$, for ζ in a set of representatives of Γ/Λ, are disjoint and their total probability is $\gamma_O(\hat{u} - \hat{v} - (0,2))/(2xH)$, as in the proof of Lemma 17(3). The vector $\hat{u} - \hat{v} - (0,2) = (-r_\ell \sin\theta,\ r_\ell \cos\theta - 2)$ has coordinates of modulus at most $2|\theta|$ and $(2 - r_\ell) + \theta^2 \leq d_+ + \theta^2$, where $d_+ \geq 2 - \sqrt{4 - l'^2}$ is a rational upper bound. By Lemma 19, for $|\theta| \in ((i-1)\Delta_\theta, i\Delta_\theta]$ with $\Delta_\theta = \eta/N_\theta$, the conditional probability is at least $\Omega(2\theta_i, d_+ + \theta_i^2)$. Integrating over $|\theta| \leq \eta$, the probability is at least

$$\frac{\Delta_\theta}{\pi} \sum_{i=1}^{N_\theta} \Omega\big(2\theta_i, d_+ + \theta_i^2\big),$$

and the program verifies that twice this, with π replaced by π_{hi}, is at least $I - J(l - \delta)$ on every medium interval. The smallest margin is $0.1744 \cdot 10^{-9}$ for the coarse and $0.0405 \cdot 10^{-9}$ for the fine profile. Since $z_u \geq \ell > l$ and $\ell > 0.02 > \delta$, the claim follows.

Small chord. The set $\mathcal{R}_\theta$ lies in $[-0.004, 0.004] \times [-0.0091, 0.0091]$, so its translates by Γ are disjoint and, for fixed θ, $\Pr_\tau(\mathcal{E}_o) = \mathrm{area}(\mathcal{R}_\theta)/(2xH) = \sum_k 4w_{i,k}\Delta_y/(2xH)$. Integrating over $|\theta| \leq \eta$, the probability is at least

$$\mathcal{P} = \frac{4\Delta_\theta \Delta_y}{\pi \cdot 2xH} \sum_{i=1}^{N_\theta} \sum_{k=0}^{N_y - 1} w_{i,k}.$$

The program computes $2\mathcal{P}$, with π replaced by π_{hi} and each W rounded in the safe direction, and finds the following lower bounds, in units of 10^{-9}: for the coarse profile 23.2555 on $[\delta, 0.02]$ and 7.9986 on $[\delta_R, \delta)$; for the fine profile 25.9743 on both intervals in $[\delta, 0.02]$ and 9.3028 on both intervals in $[\delta_R, \delta)$. These exceed I and I_R respectively, and $z_u \geq \ell$ gives the claim. □

8.3 Budgets

Call $b_R \geq 0$ *admissible* if at most b_R doubled pairs have chord less than δ_R and at most $2b_R$ arcs have chord less than δ_R; define admissibility of b_D in the same way with δ in place of δ_R.

Lemma 50. *Each of $L_{\mathcal{R}}$, $5s + \binom{s}{2}$ and $\frac{1}{2}(L_{\mathcal{R}} + 5s)$ is an admissible b_R. Each of $L_{\mathcal{D}}$, $14s + 8j + \binom{s}{2}$ and $\frac{1}{2}(L_{\mathcal{D}} + 5h)$ is an admissible b_D.*

Proof. We treat b_D; the proof for b_R is the same with $\mathcal{R}, \delta_R, s, L_{\mathcal{R}}$ in place of $\mathcal{D}, \delta, h, L_{\mathcal{D}}$, using Lemma 33(2) in place of (3). A vertex x is an endpoint of exactly $2d_x$ arcs, two on each circle S_y with y a neighbour of x, and a doubled pair with endpoint x uses two of them. An arc or doubled pair of chord less than δ has an endpoint outside $\mathcal{D}$. Hence at most $2L_{\mathcal{D}}$ such arcs and at most $L_{\mathcal{D}}$ such pairs exist. By Lemma 33(3) at most $14s+8j+\binom{s}{2}$ pairs of points are at distance less than δ, and each carries at most two arcs and at most one doubled pair. Finally, give each pair of points at distance less than δ the weight $\frac{1}{2}$ times the number of arcs joining it. The total weight bounds both the number of such doubled pairs and half the number of such arcs. If x_1 is the total weight of pairs with exactly one point outside $\mathcal{D}$ and x_2 that of pairs with both points outside $\mathcal{D}$, then counting arc endpoints outside $\mathcal{D}$ gives $2x_1 + 4x_2 \leq 2L_{\mathcal{D}}$, and Lemma 33(1) gives $x_1 \leq 5h$. So the total weight is $x_1 + x_2 \leq \frac{1}{2}(x_1 + 2x_2) + \frac{1}{2}x_1 \leq \frac{1}{2}(L_{\mathcal{D}} + 5h)$. □

Let $\tau_0 = \delta$ and $\tau_i/\delta = 1.4, 1.9, 2.4, 2.9, 3.4, 3.9, 4.4$ for $i = 1, \dots, 7$. Put $K_i = \lfloor (1 + 2\tau_i/\delta)^2 \rfloor$,

$$B_i = \left\lfloor \frac{(K_i - 1)(n - h)}{2} \right\rfloor + \min\left(2L_{\mathcal{D}},\ K_i h + \binom{h}{2}\right), \qquad \varsigma_i = 2m - 2t - B_i,$$

and, for admissible b_R, b_D,

$$Q = \pi n - (m - b_R)\delta_R - (m - b_D)(\delta - \delta_R), \qquad Q_{\mathcal{D}} = Q - \pi h + \tfrac{1}{2}\delta L_{\mathcal{D}},$$

$$Q^* = \min\left(Q - \tfrac{1}{2}\sum_{i=1}^{7}(\tau_i - \tau_{i-1})\,[\varsigma_i]_+\,,\ Q_{\mathcal{D}} - \tfrac{1}{2}\sum_{i=1}^{7}(\tau_i - \tau_{i-1})\,[\varsigma_i - L_{\mathcal{D}}]_+\right).$$

Lemma 51 (Budgets)**.** *Let $\mathcal{O}$ be the set of orientations (v, u) of doubled pairs with $u \in \mathcal{D}$ and chord $\ell \geq \delta_R$, and $\mathcal{O}_\delta \subseteq \mathcal{O}$ those with $\ell \geq \delta$. For $o \in \mathcal{O}$ let z_o be the angle of the arc of S_u joining the endpoints of the pair, and $\delta(o) = \delta$ if $\ell \geq \delta$ and δ_R otherwise. Then*

$$\tfrac{1}{2}|\mathcal{O}| \geq t - b_R - \tfrac{1}{2}L_{\mathcal{D}}, \qquad \tfrac{1}{2}|\mathcal{O}_\delta| \geq t - b_D - \tfrac{1}{2}L_{\mathcal{D}},$$
$$\tfrac{1}{2}\sum_{o \in \mathcal{O}}\big(z_o - \delta(o)\big) \leq Q^* \leq \min(Q, Q_{\mathcal{D}}).$$

Proof. Each doubled pair has two orientations. At most b_R pairs have chord less than δ_R. A vertex $u \notin \mathcal{D}$ is a centre of at most d_u doubled pairs, since each uses a different arc of S_u, so at most $L_{\mathcal{D}}$ orientations have head outside $\mathcal{D}$. This gives the first inequality, and the second in the same way.

Distinct $o \in \mathcal{O}$ correspond to distinct arcs, since the arc of S_u determines the pair and the head. Give each arc $\mathfrak{a}$ the *reserve* $\nu(\mathfrak{a}) = \delta_R \mathbf{1}[\text{chord} \geq \delta_R] + (\delta - \delta_R)\mathbf{1}[\text{chord} \geq \delta]$. Then $z(\mathfrak{a}) \geq \text{chord} \geq \nu(\mathfrak{a})$, and $\nu = \delta(o)$ on the arc of $o \in \mathcal{O}$. Call an arc *single* if its endpoints do not form a doubled pair; arcs of orientations in $\mathcal{O}$ are not single. If a single arc has chord at least τ_I, then $z - \nu \geq \tau_I - \delta = \sum_{i \leq I} (\tau_i - \tau_{i-1})$. Let $\mathcal{S}_i$ be the set of single arcs with chord at least τ_i. Since all the quantities $z - \nu$ are nonnegative, for any set $\mathcal{Z}$ of centres,

$$\sum_{\substack{o \in \mathcal{O} \\ \text{head in } \mathcal{Z}}} (z_o - \delta(o)) \leq \sum_{\mathfrak{a} \text{ on } S_u,\, u \in \mathcal{Z}} \big(z(\mathfrak{a}) - \nu(\mathfrak{a})\big) - \sum_{i=1}^{7} (\tau_i - \tau_{i-1})\, \#\{\mathfrak{a} \in \mathcal{S}_i \text{ on } S_u,\ u \in \mathcal{Z}\}.$$

The angles of the arcs on one circle add up to 2π.

A single arc with chord less than τ_i joins a pair of points at distance less than τ_i that carries no other arc. Points of $\mathcal{D}$ are more than δ apart, so the open discs of radius $\delta/2$ about them are disjoint, and at most K_i points of $\mathcal{D}$ lie within distance τ_i of any point. Hence at most $\lfloor (K_i - 1)(n - h)/2 \rfloor$ such pairs lie in $\mathcal{D}$, and at most $K_i h + \binom{h}{2}$ have a point outside $\mathcal{D}$; also at most $2L_{\mathcal{D}}$ arcs have an endpoint outside $\mathcal{D}$. So at most B_i single arcs have chord less than τ_i, and $|\mathcal{S}_i| \geq [2m - 2t - B_i]_+ = [\varsigma_i]_+$, because $2t$ of the $2m$ arcs are not single.

Take $\mathcal{Z} = X$. At most $2b_R$ arcs have chord less than δ_R and at most $2b_D$ have chord less than δ, so $\sum \nu \geq (2m - 2b_R)\delta_R + (2m - 2b_D)(\delta - \delta_R)$, and the right side above is at most $2Q - \sum_i (\tau_i - \tau_{i-1})\, [\varsigma_i]_+$. Take $\mathcal{Z} = \mathcal{D}$. There are $2m - L_{\mathcal{D}}$ arcs on circles centred in $\mathcal{D}$, the sum of their angles is at most $2\pi(n - h)$, their reserves sum to at least $(2m - L_{\mathcal{D}} - 2b_R)\delta_R + (2m - L_{\mathcal{D}} - 2b_D)(\delta - \delta_R)$, and at most $L_{\mathcal{D}}$ arcs of $\mathcal{S}_i$ lie on circles centred outside $\mathcal{D}$. So the right side is at most $2Q_{\mathcal{D}} - \sum_i (\tau_i - \tau_{i-1})\, [\varsigma_i - L_{\mathcal{D}}]_+$. □

Proposition 52 (Centre credit)**.** *Let $\mathcal{C}_{\text{ctr}}$ be the expected number of centre candidate edges o with $\tau \in \mathcal{E}_o$. For each profile and all admissible b_R, b_D,*

$$\mathcal{C}_{\text{ctr}} \geq \max\big(0,\ \mathcal{C}_1,\ \mathcal{C}_2\big),$$

where

$$\mathcal{C}_1 = I_R\left[t - b_R - \tfrac{1}{2}L_{\mathcal{D}}\right]_+ + (I - I_R)\left[t - b_D - \tfrac{1}{2}L_{\mathcal{D}}\right]_+ - J\min(Q, Q_{\mathcal{D}}),$$
$$\mathcal{C}_2 = I_R\big(t - b_R - \tfrac{1}{2}L_{\mathcal{D}}\big) + (I - I_R)\big(t - b_D - \tfrac{1}{2}L_{\mathcal{D}}\big) - JQ^*.$$

Proof. By Lemma 49, and since probabilities are nonnegative,

$$\mathcal{C}_{\mathrm{ctr}} \geq \sum_{o\in\mathcal{O}} \tfrac{1}{2}\big(I_{\delta(o)} - J(z_o - \delta(o))\big) = \tfrac{1}{2}I_R|\mathcal{O}| + \tfrac{1}{2}(I - I_R)|\mathcal{O}_\delta| - \tfrac{J}{2}\sum_{o\in\mathcal{O}}\big(z_o - \delta(o)\big),$$

where $I_\delta = I$ and $I_{\delta_R} = I_R$. Apply Lemma 51, and $|\mathcal{O}|, |\mathcal{O}_\delta| \geq 0$ for $\mathcal{C}_1$. □

Monotonicity. Both $\mathcal{C}_1$ and $\mathcal{C}_2$ are nonincreasing in b_R, b_D and $L_{\mathcal{D}}$, nondecreasing in m when t is fixed, and nondecreasing in t. For $\mathcal{C}_2$ the last claim holds because each $[\varsigma_i]_+$ and $[\varsigma_i - L_{\mathcal{D}}]_+$ decreases at rate at most 2 in t, so Q^* increases at rate at most $\tau_7 - \tau_0 = 3.4\,\delta$, and $I - 3.4\,\delta J > 0$ for both profiles. The quantity h enters through $-\pi h$, which is decreasing, and through B_i, where we may use a lower bound for h in the first term and an upper bound in the second. Consequently, if t, m and h are replaced in $\mathcal{C}_1$ and $\mathcal{C}_2$ by lower bounds, and $L_{\mathcal{D}}$, b_R, b_D and the h in $K_i h + \binom{h}{2}$ by upper bounds, the result is still a lower bound for $\mathcal{C}_{\mathrm{ctr}}$. By Lemma 43, $L_{\mathcal{R}} \leq \Lambda_{1,0}(m, s, 0)$ and $L_{\mathcal{D}} \leq \Lambda_{1,0}(m, h, 0)$, and by Lemma 45, $t \geq 2m - A^*(m)$.

We define $\mathcal{C}(m, s, j)$ to be the larger, over the two profiles, of $\max(0, \mathcal{C}_1, \mathcal{C}_2)$, evaluated with $t = 2m - A^*(m)$, $h = s + j$, $L_{\mathcal{R}} = \Lambda_{1,0}(m, s, 0)$, $L_{\mathcal{D}} = \Lambda_{1,0}(m, h, 0)$, $b_R = \min\big(L_{\mathcal{R}}, 5s + \binom{s}{2}, \frac{1}{2}(L_{\mathcal{R}} + 5s)\big)$ and $b_D = \min\big(L_{\mathcal{D}}, 14s + 8j + \binom{s}{2}, \frac{1}{2}(L_{\mathcal{D}} + 5h)\big)$. Then $\mathcal{C}_{\mathrm{ctr}} \geq \mathcal{C}(m, s, j)$, as used in Proposition 44.

9 The certificate: proof of Theorem 1

9.1 The wave datum W_V

Let $x_V = 923111560246/10^{12}$. The profile ℓ_V is symmetric about $H/2$, and on $[0, H/2]$ it is the piecewise-linear function through the following points (both coordinates in units of 10^{-12}), and 0 on $[0.00918, H/2]$:

y	$\ell_V(y)$	y	$\ell_V(y)$	y	$\ell_V(y)$
0	3673765364	3240000000	2383876640	6480000000	975983025
540000000	3465107313	3780000000	2150249605	7020000000	741894261
1080000000	3256662786	4320000000	1916357203	7560000000	509856145
1620000000	3047757021	4860000000	1681629236	8100000000	302302446
2160000000	2838089895	5400000000	1446502958	8640000000	25666952
2700000000	2618266220	5940000000	1211236622	9180000000	0

So $y_0 = 0.00918$, and the hole O is a region of width 0.00735 and height 0.01836, symmetric in both axes. The profile r_V has knots at $kH/512$, $0 \le k \le 512$, with values in $[0.874850, 0.971373]$, published with the programs (Appendix A). A profile checker verifies (W1)–(W4) exactly at the knots, including that ℓ_V is nonincreasing on $[0, H/2]$ and that its first zero y_0 satisfies the strict inequality $y_0 < H/4$ of (W3); since the profiles are piecewise linear with these knots, this gives (W1)–(W4). The program for the star certificate of W_V checks again that ℓ_V is nonincreasing and checks that r_V is nondecreasing on $[0, H/2]$, as used in Section 3.2.

By Lemma 8, and by Proposition 12 with $N = 2^{26}$,

$$q = \frac{82102466727}{1153889450307500} \approx 7.11528 \times 10^{-5},$$
$$\bar{p} \le p = \frac{102782566719698111}{145001902327661557000000} \approx 7.08836 \times 10^{-7}.$$

In the finite check below, q and p are rounded up to multiples of 10^{-18}.

9.2 Certified constants

The quantities of Section 7 are computed in exact rational arithmetic from ℓ_V alone. In lower bounds $1/\pi$ is replaced by $1/\pi_{\mathrm{hi}}$, and in upper bounds by $1/\pi_{\mathrm{lo}}$. The results are rounded down (for B, C, E) or up (for L, T, U, J) to multiples of 10^{-18}. In units of 10^{-18} the values are:

Constant	Value	Definition
B	112668096940	$\lfloor \frac{2\Delta}{\pi_{\mathrm{hi}}} \sum_k h_k \rfloor_{18}$
C	16034576942481	$\lfloor$right side of Lemma 39, $\lambda = 1\rfloor_{18}$
E	3057937052494	$\lfloor$right side of Lemma 39, $\lambda = 2\rfloor_{18}$
L	142407225756	$\lceil \frac{\Delta}{\pi_{\mathrm{lo}}} \sum_k g_a(t_k) h_k \rceil_{18}$
T	95763744940	$\lceil \frac{\Delta}{\pi_{\mathrm{lo}}} \sum_{t_k > d} g_a(t_k) h_k \rceil_{18}$
U	104770006866	$\lceil \frac{2\Delta}{\pi_{\mathrm{lo}}} \sum_{t_k \le d} k_a(t_k) h_k \rceil_{18}$
J	300592212470	$\lceil \frac{2\Delta}{\pi_{\mathrm{lo}}} \sum_k k_a(t_k) h_k \rceil_{18}$

The short-credit caps are $M_1 = \frac{1206695810097}{46155578012300000} \approx 2.6144 \times 10^{-5}$ and $M_2 = \frac{1670178375963}{230777890061500000} \approx 7.2372 \times 10^{-6}$. The retention ratios, computed exactly and rounded down to thousandths, satisfy $\rho_1 \geq 0.344$ and $\rho_2 \geq 0.761$. We use these lower bounds, so

$$\alpha = 0.828, \qquad \beta_{\mathcal{R}} = 0.1195, \qquad \beta_{\mathcal{D}} = 0.6195.$$

The program also checks the two inequalities in the proof of Lemma 36, and those of Lemma 33. With β_V from Proposition 25, $p' = p - \beta_V/2$ rounded up to a multiple of 10^{-18} is $708385488148 \cdot 10^{-18}$, and q rounded up is $71152801254159 \cdot 10^{-18}$.

9.3 The finite check

Proposition 53. *Let $n = 10{,}689$ and $m^* = 403{,}078$, with the constants above. For all integers $m \in [0, m^*]$ and $s, j \geq 0$ with $s + j \leq n$, we have $\Psi(m, s, j) \leq 0.999999995095531514 < 1$.*

Proof. Let $\mathfrak{B} = [m_0, m_1] \times [s_0, s_1] \times [j_0, j_1]$ be an integer box with $s_1 \leq n - j_0$ and $j_1 \leq n - s_0$. Each term of the three bounds in Ψ is bounded on $\mathfrak{B}$ by its value at a suitable corner, or by a combination of corner values: $m(p' - B)$ increases in m because $p' > B$; V decreases in m; $\frac{2m}{n}(\alpha s + \beta j)$ increases in all three variables; $-sC - jE$ decreases in s and j; $(n - s)L$ and $(n - s)T$ decrease in s; jU and jJ increase in j. The variance expression under the square root is not monotone (for $j = 0$ it equals $\alpha^2 s(1 - s/n)$, which increases and then decreases in s), so we bound its positive part and its subtracted square separately:

$$\alpha^2 s + \beta^2 j - \frac{(\alpha s + \beta j)^2}{n} \leq \alpha^2 s_1 + \beta^2 j_1 - \frac{(\alpha s_0 + \beta j_0)^2}{n}.$$

Hence the minimum in Ψ is at most $\bar{\Psi}(\mathfrak{B})$, the minimum of $nq + m_1 p'$ and of the two mode bounds evaluated with m_1 in $m(p' - B)$ and in the mean term, with $V(m_0)$ and the right side of the last display under the square root, with s_1, j_1 in the mean term, with s_0, j_0 in $-sC - jE$, and with s_0 in $(n - s)L$ and $(n - s)T$ and j_1 in jU and jJ.

For the centre credit, the monotonicity statement after Proposition 52 shows that $\mathcal{C}(m, s, j) \geq \underline{\mathcal{C}}(\mathfrak{B})$, where $\underline{\mathcal{C}}(\mathfrak{B})$ is computed as $\mathcal{C}$ but with: m_0 in place of m; $t = 2m_0 - A^*_{\mathfrak{B}}$, where $A^*_{\mathfrak{B}}$ is A^* with $n(n-1) + m - 2m^2/n$ replaced by its maximum over $[m_0, m_1]$, bounded by $n(n-1) + m' - 2m_0^2/n$ with $m' = m_0$ if $4m_0 \geq n$ and $m' = m_1$ otherwise; the lower bound $s_0 + j_0$

for h in $-\pi h$ and in $(K_i-1)(n-h)$, and the upper bound $\min(s_1+j_1,n)$ in $K_i h+\binom{h}{2}$, in $\frac{1}{2}(L_{\mathcal{D}}+5h)$ and in the degree bound; the upper bounds

$$\left\lceil \frac{2m_1\bar{h}}{n} \right\rceil + 1 + \left\lfloor \sqrt{2V(m_0)\left(n\bar{h}-\underline{h}^2\right)/n} \right\rfloor$$

for $L_{\mathcal{D}}$ (with $(\underline{h},\bar{h})=(s_0+j_0,\min(s_1+j_1,n))$) and for $L_{\mathcal{R}}$ (with $(\underline{h},\bar{h})=(s_0,s_1)$); s_1, j_1 in the packing bounds for b_R, b_D; and π_{hi} in place of π. The program evaluates $\bar{\Psi}(\mathfrak{B})-\underline{\mathcal{C}}(\mathfrak{B})$ in exact rational and integer arithmetic in units of 10^{-18}, rounding the mean terms up, taking the integer square root of the rounded-down radicand and adding one unit, and rounding every other term in the safe direction.

Starting from $[0,m^*]\times[0,n]\times[0,n]$, the program discards boxes with $s_0+j_0>n$, accepts a box when $\bar{\Psi}(\mathfrak{B})-\underline{\mathcal{C}}(\mathfrak{B})<1$, and otherwise bisects the side of greatest weighted length (weights $1,12,12$). It reports failure if a box consisting of a single point is not accepted. It accepts all boxes after examining $66{,}601$ of them, and the largest accepted value is $999999995095531514\cdot 10^{-18}$. This program alone proves the stated upper bound 0.999999995095531514.

A second program checks the weaker statement $\Psi(m,s,j)<1$ on the same range, which is all that the proof of Theorem 1 uses. It uses a different subdivision (cut points at two fifths of the side, weights $1,19,9$). Instead of integer square roots it compares squares exactly: writing a mode bound on a box as $A_0+B\sqrt{Q_0}$, it accepts that mode when $1-A_0>0$ and $(1-A_0)^2>B^2Q_0$, after clearing denominators. It recomputes m^* by bisection on V and derives the coefficients of V separately. It shares with the first program only the routines that evaluate $\underline{\mathcal{C}}(\mathfrak{B})$ from its inputs and the packing bounds for b_R, b_D. It accepts after examining $46{,}865$ boxes. □

9.4 Proof of Theorem 1

Suppose that a set X_0 of at most $10{,}689$ points has no proper 6-colouring of its unit-distance graph. Choose $Y\subseteq X_0$ minimal with this property. Every $v\in Y$ has at least six unit neighbours in Y, since otherwise a proper 6-colouring of $Y\setminus\{v\}$ would extend to Y. Add to Y points far from Y and from each other, so as to obtain a set X of exactly $n=10{,}689$ points whose unit-distance graph is that of Y together with isolated vertices. Then X has no proper 6-colouring, and every non-isolated vertex of X has degree at least 6, so Corollary 30 gives $m\le m^*$. By Proposition 44, $1\le\Psi(m,s,j)$, which contradicts Proposition 53. □

10 Remarks

1. At $n = 10{,}690$ the edge ceiling becomes $403{,}129$, and with the same constants the box search fails at $(m, s, j) = (401216, 173, 105)$, where the box bound is $1.0000000493\ldots$; the second program fails at a different point. So these parameters stop at $10{,}689$. The failures at $10{,}690$ only mark where this certificate stops; they do not exhibit a unit-distance graph that needs seven colours.
2. Without the forest, the star credit and the centre edges, the available upper estimate is $10{,}689\,q + 403{,}078\,p \approx 1.0463 > 1$. The star credit lowers this by only $403{,}078 \cdot \beta_V/2 \approx 1.8 \times 10^{-4}$; the decisive quantities are the unit credit $B \approx 1.127 \times 10^{-7}$, the collision losses, and the centre credit: at $m = 403{,}078$ and $s = j = 0$, Lemma 45 gives at least $166{,}729$ doubled pairs and $\mathcal{C} \approx 1.24 \times 10^{-3}$. The short credits matter for configurations with many close pairs.
3. Choosing the forest after the placement is what allows Lemma 36: for a fixed rotation, at most one of the unit edges entering a head from $\mathcal{D}$ can have both ends blank, so they do not compete. With a forest fixed before the translation, the collision loss between such tails would remain.
4. The colourings of Theorem 3 and of Section 9 have the lattice structure of Parts's construction [12]. They differ in the profiles, and in being allowed monochromatic unit pairs.
5. Both q and p are fixed positive numbers, so the method gives no colouring of the whole plane. If a sequence of periodic colourings had blank densities and monochromatic probabilities both tending to 0, a compactness argument would give $\chi(\mathbb{R}^2) \leq 6$.

A Data and programs

The profiles of W_E and W_V and the programs are available at `https://github.com/charlieoneill11/maths-releases`. The programs are written in Python, using the python-flint library for ball arithmetic, and in C++. They carry out the following checks.

- The quadrature of Corollary 16 for W_E, which verifies Theorem 3.

- The star certificates of Sections 3.3 and 3.4, and the bound $245{,}510$ of Theorem 2. The program for W_E exits with an error if the certified edge bound is below $245{,}510$.
- Proposition 12, in two independent implementations, one in integer arithmetic and one in Python, the second of which also computes q; and (W1)–(W4) for W_V, with q and p for W_V recomputed with $N = 2^{26}$.
- The lower enclosure Ω of Definition 18, the packing bounds k_ϱ and g_ϱ, and the constants of Section 9.
- Lemma 33, the probability integrations of Lemma 49, and the evaluation of $\underline{\mathcal{C}}(\mathfrak{B})$.
- The two box checks of Proposition 53 at $n = 10{,}689$. The program also reports the failures at $n = 10{,}690$, where the certificate stops, and exits with an error unless both checks pass at $n = 10{,}689$.

All of these were re-run on 24 November 2026 and reproduced their recorded outputs exactly. On one core of a 2.1 GHz Xeon, the quadrature takes about 1.5 seconds and the star certificate for W_E about 75 seconds; each of the other checks takes between 2 and 15 seconds.

B Formal verification in Lean

Theorem 2 has also been proved in the Lean 4 proof assistant [8] with the Mathlib library [10]. So has Theorem 1, from four published facts about crossing numbers that enter as an explicit hypothesis. The development uses version 4.29.1 of Lean and the matching release of Mathlib, and is available at `https://github.com/charlieoneill11/maths-releases`.

Statements. In the formal statements the plane is Mathlib's Euclidean space $\mathbb{R}^2$, and a colouring is any function from the plane to a set of six colours. Lean proves the following.

- If E is a finite set of pairs (u, v) of points with $|u - v| = 1$, and E has at most $245{,}510$ elements, then some colouring c has $c(u) \neq c(v)$ for every $(u, v) \in E$. Listing each edge in one orientation, this is Theorem 2.
- Given the hypothesis described below, if X is a set of at most $10{,}689$ points, then some colouring c has $c(x) \neq c(y)$ whenever $x, y \in X$ and $|x - y| = 1$. This is Theorem 1.

The formal statements also define graphs, drawings by continuous injective arcs, good drawings, crossing counts and the crossing number. The hypoth-

esis consists of four facts. Two are the inequality $\mathrm{cr}(G) \geq 5M - \frac{203}{9}(N-2)$ of [3, Theorem 4(b)] and the density theorem [4, Theorem 7]. The other two are the facts about drawings from [14] used in Section 5: $\mathrm{cr}(G) \leq \mathrm{cr}(\mathcal{D})$ for every drawing $\mathcal{D}$ in which any two edges have finitely many common points, and some good drawing of G has exactly $\mathrm{cr}(G)$ crossings. From these, Lean proves Theorem 26 and the rest of Section 5. Theorem 2 does not use them. The formal statements were reviewed against the paper and then kept unchanged.

Proof. Every result of Sections 2 to 9 has a Lean counterpart. The Lean proof follows the paper with two changes. Each is easier to formalise and gives a slightly weaker intermediate constant that still suffices.

- The height quadrature of Section 2 (Proposition 13 to Corollary 16) is not formalised. For W_E, Lean instead evaluates the bound of Proposition 12 with $N = 2^{28}$ and proves that a randomly placed unit segment is monochromatic with probability at most $B = 4077152 \cdot 10^{-12}$. This is larger than p_E, so the constant of Theorem 3 itself is not formalised. The argument of Section 4 goes through with B in place of p_E, since $245{,}269\, B < 1$ and $245{,}510\,(B - \beta_E/2) = 0.99999954752 < 1$.

- Lemma 32 is not used. Both collision parameters of Section 7 use the pointwise bound k_ϱ of Lemma 31 in place of g_ϱ. This raises the constants L and T of Section 9 to 150296106235 and 97911102803, in the units used there, and the box search of Proposition 53, which Lean also carries out, still accepts every feasible parameter triple at $n = 10{,}689$.

Every other numerical input is recomputed by a checker written in Lean: Proposition 12 for W_V, the star certificates of Proposition 25 for both colourings, the constants of Section 9 and the integrations of Section 8. The recomputed blank density and monochromatic probability of W_V, and the constants B, C, E, U and J of Section 9, agree exactly with the values printed here. The programs of Appendix A are used only for comparison.

Trust. The checkers are evaluated with Lean's `native_decide` tactic, which runs compiled code, so accepting the formal proof means trusting the Lean compiler for these evaluations as well as the Lean kernel. The proof of Theorem 2 uses 8 such evaluations: six groups of angular bins for Proposition 12 and two checks for the star certificate. The proof of Theorem 1 uses 17 others: six groups of angular bins for Proposition 12, five checks for the star certificate of W_V, two for the constants of Section 9, three for the integrations of Section 8, and the box search of Proposition 53. Apart from these evaluations, the proofs use only the three standard axioms of Lean: propositional extensionality, the axiom of choice and the soundness of quotients. They contain no unproved steps, declare no further axioms and

use no floating-point arithmetic. A clean build of the development takes about 14 minutes on a cloud machine provided by Blaxel.

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