A prime between every two consecutive $\frac{11}{5}$-th powers

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Abstract

Legendre's conjecture asserts that there is a prime between any two consecutive squares. A natural weakening replaces the exponent 2 by $\alpha > 2$. Chamberland and Straub showed, assuming the Riemann hypothesis, that the case $\alpha = 11/5$ reduces to a finite verification, and the best published statement valid for every positive integer is at $\alpha = 2.2253$. We prove that the interval $(n^{11/5}, (n+1)^{11/5})$ contains a prime for every positive integer n. The proof uses the Riemann hypothesis, which Alpöge has proved, so the result is unconditional. An explicit-formula argument with a compactly supported test function of twelve cosine modes, together with the Brun–Titchmarsh inequality of Montgomery and Vaughan and the theorem of Carneiro, Milinovich and Soundararajan, handles every real $x \geq 3.5 \times 10^{13}$. This part has also been proved in Lean 4 with Mathlib, from the Riemann hypothesis and five published results taken as hypotheses. Below that we use a computation. It certifies a prime in $(m^2, (m+1)^2)$ for every $m \leq 10^{14}$, which extends the verification of Legendre's conjecture by Sorenson and Webster by about 42%, and a prime in $(n^{11/5}, (n+1)^{11/5})$ for each of the remaining 2.97×10^{13} bases $n < 3.5 \times 10^{13}$. Its archive is held in private storage. The certificates released with the paper, with the computation of Sorenson and Webster, cover every $n \leq 3{,}883{,}865{,}444{,}451$ and every $m \leq 70{,}500{,}002{,}100{,}000$; for the other bases the theorems rest on the verification done during the original run, of which 1.6×10^8 checks have been replayed independently. A final section lowers the analytic cutoff to $32{,}746{,}701{,}927{,}355$, using a weighted large sieve over Farey fractions with squarefree denominators and lower bounds for the number of semiprimes obtained from the complete prime-gap histogram below 4×10^{18}. With the first 1.4×10^6 zeros of ζ and the table of maximal prime gaps up to 1.32×10^{20}, whose completeness rests on an exhaustive search not published in a refereed journal, the cutoff becomes $32{,}654{,}578{,}787{,}390$.

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AI use and author contributions

The mathematical arguments and computer programs in this paper were developed by Astra (GPT-6.2 Astra, OpenAI). Fable (Fable 5.2, Anthropic) worked throughout as a second model: it proposed alternative approaches, reviewed arguments and raised objections. The manuscript was prepared from the project's research records with Claude (Anthropic). All computations were re-run before release, and the programs and certificates are available at https://github.com/charlieoneill11/maths-releases.

Charles O'Neill proposed the problem because the Riemann hypothesis, now a theorem [1], reduces the exponent $\frac{11}{5}$ to a finite verification, as Chamberland and Straub observed [5]. His technical contribution was to propose closing the finite range by direct computation, and to provide the cloud computing resources for the verification described in Section 5. Charles O'Neill has checked the arguments and takes full responsibility for the correctness of the results.

1 Introduction

Legendre's conjecture, that $(m^2, (m+1)^2)$ contains a prime for every integer $m \geq 1$, remains open even though the Riemann hypothesis is now a theorem [1]. Under RH there is a prime in every interval $[y, y + c\sqrt{y}\log y]$ with y large [6, 4]; the interval $(m^2, (m+1)^2)$ has length $2m + 1 \asymp \sqrt{y}$, which is slightly too short. For an exponent $\alpha > 2$ the interval $(n^\alpha, (n+1)^\alpha)$ has length about $\alpha n^{\alpha-1} = \alpha y^{1-1/\alpha}$ at height $y = n^\alpha$, which exceeds $\sqrt{y}\log y$ for large n. Unconditionally, Baker, Harman and Pintz [2] give a prime in $[y - y^{0.525}, y]$ for large y, so for every $\alpha > 1/0.475 \approx 2.105$ the interval $(n^\alpha, (n+1)^\alpha)$ contains a prime for all sufficiently large n. That theorem gives no explicit starting point.

Carneiro, Milinovich and Soundararajan [4] proved, assuming RH, an explicit version with constant $\frac{22}{25}$. Chamberland and Straub [5] used it to study $(n^\alpha, (n+1)^\alpha)$ for every n. Their argument proves the statement for all real $x \geq 1$ when $\alpha \geq 2.2253$, and at $\alpha = 11/5$ it leaves a finite range of integers. By Alpöge's theorem [1] these results are unconditional, and we use RH freely, citing [1] where it enters. It enters only through Lemmas 10 and 14, and hence through Theorems 4 and 6 and Theorem 1; Theorem 2 and the finite computation do not use it, and GRH is not used. We found no published statement valid for every positive integer at exponent 2.2.

Theorem 1. *For every positive integer n there is a prime p with*

$$n^{11/5} < p < (n+1)^{11/5}.$$

The computation behind Theorem 1 also extends the verification of Legendre's conjecture. Sorenson and Webster [20] verified it for $m \le 7.05 \times 10^{13}$.

Theorem 2. *For every integer m with $1 \le m \le 10^{14}$ there is a prime p with $m^2 < p < (m+1)^2$.*

The proof of Theorem 1 has three parts. Throughout, $\alpha = 11/5$ and

$$n_1 = 3{,}883{,}853{,}444{,}450.$$

Proposition 3 (Low range)**.** *Theorem 1 holds for $1 \le n \le n_1$.*

Theorem 4 (Large range)**.** *For every real $x \ge 35 \cdot 10^{12}$ the open interval $(x^{11/5}, (x+1)^{11/5})$ contains a prime.*

Proposition 5 (Finite range)**.** *For every integer n with $n_1 < n < 35 \cdot 10^{12}$ there is a prime p with $n^{11} < p^5 < (n+1)^{11}$.*

Proof of Theorem 1. By Lemma 7 below, $n^{11} < p^5 < (n+1)^{11}$ is equivalent to $p \in (n^{11/5}, (n+1)^{11/5})$. Proposition 3, Proposition 5 and Theorem 4 cover all $n \ge 1$. □

Proposition 3 follows from the computation of Sorenson and Webster (Section 2). Theorem 2 and Proposition 5 are proved by a direct computation, which exhibits a prime with a short primality certificate for each of 5.9×10^{13} bases (Section 5).

The computation was run in November 2026 on rented machines, and its archive is held in private storage. Section 5.3 separates what this run checked from what has been reproduced independently. Briefly: the certificates released with this paper, together with [20], prove Theorem 1 for every $n \le 3{,}883{,}865{,}444{,}451$ and Theorem 2 for every $m \le 70{,}500{,}002{,}100{,}000$, and anyone can replay them; they also cover some isolated ranges. For the other bases below $35 \cdot 10^{12}$, respectively up to 10^{14}, Theorems 1 and 2 rest on the verification performed during the original run and on its audit. So far a second party has replayed $160{,}000{,}000$ of its checks. Theorems 1 and 2 have therefore not been verified independently in full.

Theorem 4 is proved in Section 4 by an explicit-formula argument in the spirit of [4]. If (ae^{-D}, ae^{D}) contains no prime, then a sum over primes

weighted by a compactly supported test function is controlled by its values on the two *shoulders* just outside the interval. The explicit formula then forces an inequality between $D\sqrt{a}$ and $\log(1/D)$ that fails for large a. We use a test function with twelve cosine modes and bound the number of primes on the shoulders by the Brun–Titchmarsh inequality of Montgomery and Vaughan [12]. This gives the range $[35 \cdot 10^{12}, 6 \cdot 10^{14}]$, and the theorem of Carneiro, Milinovich and Soundararajan covers larger x (Corollary 11).

In Appendix A we show that the analytic argument can be pushed further down.

Theorem 6. *(a) For every real $x \geq 32{,}971{,}281{,}678{,}698$ the open interval $(x^{11/5}, (x+1)^{11/5})$ contains a prime. Using the prime-gap histogram below 4×10^{18} of Oliveira e Silva, Herzog and Pardi [14], the same holds for every real $x \geq 32{,}746{,}701{,}927{,}355$.*

(b) Assume, in addition, the bounds (G2) of Table 3 for the gaps between consecutive primes below 1.32×10^{20}. Then the same holds for every real $x \geq x_0 = 32{,}654{,}578{,}787{,}390$.

Theorem 6 is not used in the proof of Theorem 1. Part (a) shortens the finite range by $35 \cdot 10^{12} - 32{,}746{,}701{,}927{,}355 = 2{,}253{,}298{,}072{,}645$ bases, and part (b) by $35 \cdot 10^{12} - x_0 = 2{,}345{,}421{,}212{,}610$ bases, about 8% of the bases checked directly. The proof sharpens the count of primes on the shoulders in three ways. The first is a weighted large sieve over Farey fractions with squarefree denominators. The second is a lower bound for the number of semiprimes among the sieve's survivors: from the complete prime-gap histogram below 4×10^{18} [14] in part (a), and in part (b) also from the table of maximal prime gaps [19]. The third, on a short final range in part (b), is the signed contribution of the first 1.4×10^6 zeros [15, 10]. The table [19] has not appeared in a refereed journal, and above 4×10^{18} its use requires that the exhaustive search for maximal gaps through 1.32×10^{20} reported there be complete. This is why part (b) is stated with that assumption.

Remark (Why $11/5$). Exponents tending to 2 would give Legendre's conjecture. Suppose the statement held for a sequence $\alpha_j \downarrow 2$. Fix m and choose j with $(m+1)^{\alpha_j} < (m+1)^2 + 1$. A prime p with $m^{\alpha_j} < p < (m+1)^{\alpha_j}$ satisfies $m^2 < p \leq (m+1)^2$, and $(m+1)^2$ is not prime, so $p \in (m^2, (m+1)^2)$. A single exponent gives nothing towards Legendre's conjecture. The exponent $11/5$ is the first of the form $2 + 1/k$ at which the finite range left by the analytic argument is within reach of direct computation.

Notation

We write $\alpha = \frac{11}{5}$ and $\sigma = \frac{11}{10}$. For real $x \geq 1$ put

$$a = (x(x+1))^{\sigma}, \quad D = \sigma \log(1 + 1/x), \quad P = D\sqrt{a}, \quad L = \log \sqrt{a}, \tag{1}$$

so that $x^{\alpha} = ae^{-D}$ and $(x+1)^{\alpha} = ae^{D}$. Also $W = aD$ and $\Delta = D/(2\pi)$. The Fourier transform is $\widehat{g}(y) = \int g(s)e^{2\pi i y s}\,\mathrm{d}s$. Sums over γ run over the ordinates of the nontrivial zeros $\frac{1}{2} + i\gamma$ of ζ, with multiplicity. Rational numbers written in decimal are exact.

2 Elementary reductions

Lemma 7. *Let $n \geq 1$ be an integer. Neither n^{α} nor $(n+1)^{\alpha}$ is prime, and a prime p lies in $(n^{\alpha}, (n+1)^{\alpha})$ if and only if $n^{11} < p^5 < (n+1)^{11}$.*

Proof. If $n^{11/5} = m$ is an integer then $m^5 = n^{11}$, so $5v_{\ell}(m) = 11v_{\ell}(n)$ and $5 \mid v_{\ell}(n)$ for every prime ℓ. Hence $n = k^5$ and $m = k^{11}$, which is 1 or composite. The same applies to $n+1$. The equivalence holds because $t \mapsto t^5$ is increasing on $[0, \infty)$. □

Lemma 8. *Let $N_0 \geq 1$ be an integer such that for every integer $1 \leq m \leq N_0$ each of the closed intervals $[m^2, m(m+1)]$ and $[m(m+1), (m+1)^2]$ contains a prime. Then for every real y with $1 \leq y < N_0$ and $y^2 \notin \mathbb{Z}$ the open interval $(y^2, (y+1)^2)$ contains a prime.*

Proof. Let $m = \lfloor y \rfloor$, so $1 \leq m \leq N_0 - 1$. Suppose first that $y^2 \leq m(m+1)$, and let p be a prime in $[m(m+1), (m+1)^2]$. Then $p \geq y^2$, and $p \neq y^2$ as $y^2 \notin \mathbb{Z}$; also $p \neq (m+1)^2$, so $p < (m+1)^2 \leq (y+1)^2$. Otherwise $y^2 > m(m+1)$, and since $y \geq m$,

$$(y+1)^2 = y^2 + 2y + 1 > m(m+1) + 2m + 1 = (m+1)(m+2) - 1.$$

Let p be a prime in $[(m+1)^2, (m+1)(m+2)]$; here $m+1 \leq N_0$. Both endpoints are composite, so $y^2 < (m+1)^2 < p \leq (m+1)(m+2) - 1 < (y+1)^2$. □

Proof of Proposition 3. Sorenson and Webster [20] verified that $(m^2, m(m+1))$ and $(m(m+1), (m+1)^2)$ contain primes for $2 \leq m \leq N_0 = 7.05 \times 10^{13}$; for $m = 1$ the closed half-intervals contain 2 and 3. Let $n \geq 1$ be an integer with $n^{11} < N_0^{10}$, that is $n^{\sigma} < N_0$. Since $t \mapsto t^{\sigma}$ has derivative at least $\sigma > 1$ on $[1, \infty)$, we have $(n+1)^{\sigma} > n^{\sigma} + 1$. Choose a real y with $n^{\sigma} < y < \min(N_0, (n+1)^{\sigma} - 1)$ such that y^2 is not an integer. By Lemma 8

there is a prime in $(y^2,(y+1)^2) \subset (n^\alpha,(n+1)^\alpha)$. Exact integer arithmetic gives

$$n_1^{11} < N_0^{10} < (n_1+1)^{11},$$

which proves the proposition. □

Lemma 9. *Let $n \geq 400$ be an integer with $n^{11} \leq 10^{140}$, and let m be the least integer with $m^{10} \geq n^{11}$. Then $1 \leq m \leq 10^{14}$ and $(m^2,(m+1)^2) \subset (n^\alpha,(n+1)^\alpha)$.*

Proof. Here $m = \lceil n^{11/10} \rceil$, and $m \leq 10^{14}$ because $n^{11/10} \leq 10^{14}$. By the mean value theorem $(n+1)^{11/10} - n^{11/10} \geq \frac{11}{10} n^{1/10} \geq \frac{11}{10} \cdot 400^{1/10} > 2$, the last because $1.1^{10} \cdot 400 > 2^{10}$. Hence $n^{11/10} \leq m < n^{11/10}+1$ and $m+1 < n^{11/10}+2 < (n+1)^{11/10}$. Squaring gives $n^\alpha \leq m^2$ and $(m+1)^2 < (n+1)^\alpha$. □

Lemma 10. *Let $X < Y$ be real with $Y \geq 5$ and*

$$Y - X - \tfrac{1}{2} > \tfrac{22}{25}\sqrt{Y}\log Y.$$

Then (X,Y) contains a prime.

Proof. By RH [1] and [4, Theorem 5], for every $u \geq 4$ the closed interval $[u, u+\frac{22}{25}\sqrt{u}\log u]$ contains a prime. Suppose first that $X \geq 4$, and choose ε with $0 < \varepsilon < \frac{1}{2}$. Then $u = X+\varepsilon$ satisfies $4 \leq u < Y$, since $Y - X > \frac{1}{2}$. As $t \mapsto \sqrt{t}\log t$ is increasing on $[1,\infty)$, a prime p in $[u, u+\frac{22}{25}\sqrt{u}\log u]$ satisfies

$$X < X+\varepsilon \leq p \leq X+\varepsilon+\tfrac{22}{25}\sqrt{Y}\log Y < X+\varepsilon+Y-X-\tfrac{1}{2} < Y.$$

Suppose next that $X < 4$. If $Y > 5$ the prime 5 lies in (X,Y). If $Y = 5$, the hypothesis gives $X < \frac{9}{2} - \frac{22}{25}\sqrt{5}\log 5 < \frac{3}{2}$, since $\frac{22}{25}\sqrt{5}\log 5 > 3.16$, and the prime 3 lies in (X,Y). □

Corollary 11. *For every real $x \geq 6 \cdot 10^{14}$ the interval $(x^\alpha,(x+1)^\alpha)$ contains a prime.*

Proof. Put $X = x^\alpha$, $Y = (x+1)^\alpha$ and $\varepsilon = 2 \cdot 10^{-15}$. By the mean value theorem $Y - X \geq \alpha x^{\alpha-1} = \frac{11}{5}x^{6/5}$. Since $1.1/x < 1.9 \cdot 10^{-15}$, we have $(1+1/x)^{11/10} \leq e^{1.1/x} < 1+\varepsilon$ and $\log(1+1/x) < \varepsilon$, so $\sqrt{Y} = (x+1)^{11/10} \leq (1+\varepsilon)x^{11/10}$ and $\log Y \leq \frac{11}{5}(\log x + \varepsilon)$. Also $\frac{1}{2} < 2 \cdot 10^{-17} \cdot \frac{11}{5}x^{11/10}$. Dividing the condition of Lemma 10 by $\frac{11}{5}x^{11/10}$, it suffices that

$$x^{1/10} - \tfrac{22}{25}(1+\varepsilon)(\log x + \varepsilon) - 2 \cdot 10^{-17} > 0.$$

The derivative of the left side with respect to $\log x$ is $\frac{1}{10}x^{1/10} - \frac{22}{25}(1+\varepsilon)$, which is positive once $x^{1/10} > 8.8\,(1+\varepsilon)$. At $x = 6 \cdot 10^{14}$ we have $x^{1/10} > 30.04$, and the left side exceeds 0.1. $\square$

Remark. Chamberland and Straub [5, §3] apply [4] at $y = n^{\alpha}$. Their displayed bound starts at $921{,}755{,}345{,}005{,}754$, and evaluating their comparison [5, (6)] directly gives every integer $n \geq 571{,}412{,}224{,}574{,}286$. We do not need these values.

3 The explicit formula and a necessary condition for a gap

Definition 12. Let $K > 0$, let $J \geq 1$, and let $w_0, \dots, w_{J-1}$ be real. Put $k_j = 2j+1$ and

$$G(s) = \mathbf{1}_{[-K,K]}(s) \sum_{j=0}^{J-1} w_j \cos\Big(\frac{\pi k_j s}{2K}\Big), \qquad F = \widehat{G}, \qquad G_+ = \max(G, 0).$$

We call G a *cosine test* of support K.

Lemma 13. *Let G be a cosine test. Then G is even and Lipschitz, $G(\pm K) = 0$, $\|G\|_\infty \leq \sum |w_j|$ and $\|G\|_1 \leq 2K \sum |w_j|$. Its transform is real, even and entire, with $F(0) = \int G$,*

$$F(y) = \frac{4K}{\pi} \cos(2\pi K y) \sum_{j=0}^{J-1} \frac{(-1)^j k_j w_j}{k_j^2 - (4Ky)^2}, \qquad F\Big(\frac{k_j}{4K}\Big) = K w_j, \tag{2}$$

and $|F(y)| \ll_G (1+|y|)^{-2}$. Moreover $\|F\|_\infty \leq \|G\|_1$ and $|F'(y)| \leq 2\pi K^2 \sum |w_j|$ for all y.

Proof. Each term vanishes at $\pm K$ since k_j is odd, so G is continuous; it is piecewise smooth, hence Lipschitz. For $\omega = \pi k/(2K)$ and $\beta = 2\pi y$ with $\omega \neq \pm\beta$,

$$\begin{aligned} \int_{-K}^{K} \cos(\omega s)\cos(\beta s)\,\mathrm{d}s &= \frac{\sin((\omega-\beta)K)}{\omega-\beta} + \frac{\sin((\omega+\beta)K)}{\omega+\beta} \\ &= \frac{(-1)^{(k-1)/2}\, 2\omega \cos(\beta K)}{\omega^2 - \beta^2}, \end{aligned}$$

because $\sin(\pi k/2 \mp \beta K) = (-1)^{(k-1)/2}\cos(\beta K)$ for odd k. Since $2\omega/(\omega^2 - \beta^2) = 4Kk/(\pi(k^2-(4Ky)^2))$, this gives (2); the value at $4Ky = k_j$ follows by continuity, or directly. Each summand of (2) is $O(y^{-2})$. Finally $|F| \le \int |G|$ and $|F'(y)| \le 2\pi \int |sG(s)|\,\mathrm{d}s \le 2\pi K^2 \sum |w_j|$. □

Lemma 14 (Explicit formula)**.** *Let G be a cosine test of support K, and let $D > 0$ and a satisfy $ae^{-KD} \ge 2$. Then*

$$\sum_{m\ge 2} \frac{\Lambda(m)}{\sqrt{m}} G\Big(\frac{\log(m/a)}{D}\Big) + D\sum_{\gamma} F(\Delta\gamma)\cos(\gamma \log a) + E_0 = D\sqrt{a}\int G(s)\cosh\Big(\frac{Ds}{2}\Big)\,\mathrm{d}s, \quad (3)$$

where $\Delta = D/(2\pi)$, $E_0 = D\sum_{k\ge 1} a^{-2k-1/2}\int G(s)e^{-(2k+1/2)Ds}\,\mathrm{d}s$, and the sum over γ converges absolutely.

Proof. Put $f(t) = t^{-1/2}G(\log(t/a)/D)$ for $t > 0$. It is Lipschitz and supported in $[A, B] = [ae^{-KD}, ae^{KD}] \subset [2, \infty)$. Its Mellin transform $\tilde f(w) = \int_0^\infty f(t)t^{w-1}\,\mathrm{d}t$ is entire, and the substitution $t = ae^{D\sigma}$ gives

$$\tilde f(w) = Da^{w-1/2}\int G(\sigma)e^{(w-1/2)D\sigma}\,\mathrm{d}\sigma.$$

We use the explicit formula of von Mangoldt in the truncated form [7, Ch. 17]: for $x \ge 2$ and $T \ge 2$,

$$\psi_0(x) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \log 2\pi - \tfrac12\log(1-x^{-2}) + R(x,T),$$
$$|R(x,T)| \le C\Big(\frac{x\log^2(xT)}{T} + \log x \min\Big(1, \frac{x}{T\langle x\rangle}\Big)\Big),$$

with an absolute constant C. Here $\psi_0(x)$ is the mean of the left and right limits of ψ at x, so $\psi_0(x) = \psi(x)$ unless x is a prime power, and $\langle x\rangle$ is the distance from x to the nearest prime power other than x itself. We only use the formula for x in the support $[A, B]$ of f, where $x \ge A \ge 2$. Since f is Lipschitz with compact support in $(1, \infty)$, Stieltjes integration by parts gives $\sum_m \Lambda(m)f(m) = \int f\,\mathrm{d}\psi = -\int \psi(t)f'(t)\,\mathrm{d}t$. In the last integral we may replace ψ by ψ_0 and then by the right side of the formula, since they differ only on the countable set of prime powers. We integrate each term against $-f'$. First, $-\int tf'(t)\,\mathrm{d}t = \int f = \tilde f(1)$. Next, $\int (t^\rho/\rho)f'(t)\,\mathrm{d}t =$

$-\int t^{\rho-1}f(t)\,\mathrm{d}t = -\tilde f(\rho)$. The constant contributes $\log 2\pi\int f' = 0$. Since $\frac{d}{dt}\log(1-t^{-2}) = 2\sum_{k\ge1} t^{-2k-1}$,

$$\tfrac12\int \log(1-t^{-2})f'(t)\,\mathrm{d}t = -\sum_{k\ge1}\int t^{-2k-1}f(t)\,\mathrm{d}t = -\sum_{k\ge1}\tilde f(-2k).$$

Finally $|\int R(t,T)f'(t)\,\mathrm{d}t| \le \|f'\|_\infty \int_A^B |R(t,T)|\,\mathrm{d}t$, and we let $T\to\infty$ with f fixed. The first part of R contributes $O(B^2\log^2(BT)/T)$. For the second, $\min(1, B/(T\langle t\rangle)) \le B/T + \sum_u \min(1, B/(T|t-u|))$, the sum over the prime powers $u \ne t$ with $|t-u|\le 1$. The term B/T contributes $O(B^2\log B/T)$, and each prime power $u\in[A-1,B+1]$ contributes $O(B\log B\log T/T)$ for $T\ge B$, since $\int_{|t-u|\le1}\min(1,B/(T|t-u|))\,\mathrm{d}t \le \frac{2B}{T}(1+\log\frac{T}{B})$. All these tend to 0 as $T\to\infty$, the implied constants depending only on A, B and f. Hence

$$\sum_m \Lambda(m)f(m) = \tilde f(1) - \lim_{T\to\infty}\sum_{|\gamma|<T}\tilde f(\rho) - \sum_{k\ge1}\tilde f(-2k).$$

By RH [1] each $\rho = \frac12 + i\gamma$ with γ real, and then $\tilde f(\rho) = Da^{i\gamma}F(\Delta\gamma)$. As $|F(y)| \ll (1+|y|)^{-2}$ and the number of γ in $[T,T+1]$ is $O(\log T)$, the sum converges absolutely. The zeros are symmetric under $\gamma\mapsto-\gamma$ and F is even, so the sum equals $D\sum_\gamma F(\Delta\gamma)\cos(\gamma\log a)$. Finally $\tilde f(1) = D\sqrt{a}\int G(\sigma)e^{D\sigma/2}\,\mathrm{d}\sigma = D\sqrt{a}\int G\cosh(D\sigma/2)$ since G is even, and $\tilde f(-2k)$ is the kth term of E_0. □

Lemma 15. *For $t>0$ let $N(t)$ be the number of $\gamma\in(0,t]$, with multiplicity, and write $N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac78 + R(t)$. Then $|R(t)| \le 3 + 0.15\log^+ t$.*

Proof. Trudgian [21, Cor. 1] proved $|R(t)| \le 0.112\log t + 0.278\log\log t + 2.510 + 0.2/t$ for $t\ge e$. Writing $\ell = \log t \ge 1$, the function $0.278\log\ell - 0.038\ell$ has maximum $0.278(\log(0.278/0.038)-1) < 0.276$, so $|R(t)| \le 0.15\ell + 0.276 + 2.510 + 0.2/e < 3 + 0.15\ell$. For $0<t<e$ we have $N(t)=0$, and $\frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac78$ lies in $[-\frac18,\frac78]$ there, since its minimum over $t>0$ is $\frac78 - 1$, at $t = 2\pi$. □

Lemma 16. *Let F be real, even and C^1 with $|F(y)| \ll (1+|y|)^{-2}$, and put*

$$\|F\|_1 = \int_{\mathbb R}|F|, \qquad M_F = \int_{\mathbb R}|F(y)|\log|y|\,\mathrm{d}y,$$
$$V_0 = \int_0^\infty |F'|, \qquad V_1 = \int_0^\infty \log^+ y\,|F'(y)|\,\mathrm{d}y,$$

all finite. For $0 < D < 1$ and $\Delta = D/(2\pi)$,

$$D\sum_{\gamma}|F(\Delta\gamma)| \le \|F\|_1 \log\frac{1}{D} + M_F + 2D\|F\|_\infty \\ + 2D\Big[\Big(3 + 0.15\log\frac{2\pi}{D}\Big)V_0 + 0.15V_1\Big].$$

Proof. Put $\phi(t) = |F(\Delta t)|$. Since F is C^1 and $|\cdot|$ is 1-Lipschitz, ϕ is absolutely continuous on every bounded interval, with $|\phi'(t)| \le \Delta|F'(\Delta t)|$ almost everywhere. (We do not assume that F' is bounded.) Let $M(t) = \frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac{7}{8}$ and let $t_0 \in (9,10)$ be its zero; $N = 0$ on $(0,t_0]$, so $R(t_0) = 0$. For $T > t_0$, integrating by parts on $[t_0, T]$ with $N = M + R$ and $M'(t) = \frac{1}{2\pi}\log\frac{t}{2\pi}$,

$$\int_{t_0}^{T}\phi\,\mathrm{d}N = \phi(T)R(T) + \int_{t_0}^{T}\phi(t)\frac{1}{2\pi}\log\frac{t}{2\pi}\,\mathrm{d}t - \int_{t_0}^{T}R(t)\phi'(t)\,\mathrm{d}t.$$

As $T \to \infty$, $\phi(T)R(T) \to 0$ because $|F(y)| \ll (1+|y|)^{-2}$ and $|R(T)| \le 3 + 0.15\log T$, and the two integrals converge absolutely because M_F, $\|F\|_1$, V_0 and V_1 are finite. By symmetry of the zeros,

$$\tfrac{1}{2}\sum_{\gamma}|F(\Delta\gamma)| = \int_{t_0}^{\infty}\phi\,\mathrm{d}N = \int_{t_0}^{\infty}\phi(t)\frac{1}{2\pi}\log\frac{t}{2\pi}\,\mathrm{d}t - \int_{t_0}^{\infty}R(t)\phi'(t)\,\mathrm{d}t.$$

By Lemma 15 and $\log^+(y/\Delta) \le \log(2\pi/D) + \log^+ y$, the last integral is at most $\int_0^\infty (3 + 0.15\log^+(y/\Delta))|F'(y)|\,\mathrm{d}y \le (3 + 0.15\log\frac{2\pi}{D})V_0 + 0.15V_1$. In the first integral substitute $y = \Delta t$; since $2\pi\Delta = D$ it equals $D^{-1}\int_{\Delta t_0}^{\infty}|F(y)|\log(y/D)\,\mathrm{d}y$. The integrand is positive for $y > D$, so extending the range to $(0,\infty)$ changes the integral by at most $\int_0^D |F(y)|\log(D/y)\,\mathrm{d}y \le D\|F\|_\infty$. Multiplying by $2D$ and using that F is even gives the claim. □

Throughout, $U = 1 + 10^{-12}$.

Proposition 17 (A necessary condition for a gap)**.** *Let G be a cosine test of support $1 \le K \le \frac{5}{2}$ with $\sum|w_j| < 1$ and $\|G\|_1 < 4$, and let F, M_F, V_0, V_1 be as above. Let $a > 1$ and $D > 0$ satisfy*

$$\log a \ge 60, \qquad KD \le \frac{U-1}{U}, \qquad KP < 100, \tag{4}$$

where $P = D\sqrt{a}$, and suppose that the open interval (ae^{-D}, ae^{D}) contains no prime. (Note $(U-1)/U < 10^{-12}$, so $KD < 10^{-12}$.) Let $\mathcal{B}$ be a set of positive ordinates γ, counted once each, with $F(\Delta\gamma) \ge 0$. Put $\ell = \log a + KD$ and

$$E_{\mathrm{pow}} = \Big(2 + \frac{\ell}{\log 2}(1 + KPU)\Big)\frac{\ell U}{\sqrt{a}}.$$

Then

$$PF(0) \le \sum_{1<|s_p|\le K} \frac{\log p}{\sqrt{p}} G_+(s_p) + \|F\|_1 \log\frac{1}{D} + M_F \\ - 2D\sum_{\gamma\in\mathcal{B}} F(\Delta\gamma)\big(1-\cos(\gamma\log a)\big) + E_{\mathrm{pow}} \\ + 2D\Big[\Big(3+0.15\log\frac{2\pi}{D}\Big)V_0 + 0.15V_1 + 4\Big] + 10^{-12}, \quad (5)$$

where the sum runs over primes p and $s_p = \log(p/a)/D$.

Proof. Since $\log a \ge 60$ and $KD < 10^{-12}$, we have $ae^{-KD} > e^{59} > 2$, so Lemma 14 applies. We bound the terms of (3). Since $\cosh u \le 1+u^2$ for $|u| \le 1$, the right side of (3) is at least $PF(0) - P\|G\|_1(KD)^2/4 \ge PF(0) - 10^{-22}$, as $P < 100$. Also $|E_0| \le D\|G\|_1 \sum_{k\ge1}(ae^{-KD})^{-2k-1/2} < 10^{-20}$.

In the sum over m, write $s_m = \log(m/a)/D$; only $|s_m| \le K$ contributes. Primes with $|s_p| < 1$ do not exist by hypothesis. For primes with $1 < |s_p| \le K$ we use $G \le G_+$. The remaining terms are the at most two endpoints $ae^{\pm D}$ and the proper prime powers $m = p^r$, $r \ge 2$, in $[A,B] = [ae^{-KD}, ae^{KD}]$. Each has $\Lambda(m) \le \log B = \ell$, $m^{-1/2} \le U/\sqrt{a}$ and $|G| \le \sum|w_j| < 1$. The exponent satisfies $2 \le r \le \ell/\log 2$, so fewer than $\ell/\log 2$ exponents occur. For each r, the number of rth powers in $[A,B]$ is at most

$$1 + B^{1/r} - A^{1/r} \le 1 + \sqrt{B} - \sqrt{A} = 1 + 2\sqrt{a}\sinh(KD/2) \le 1 + KPU,$$

because $\frac{1}{r}u^{1/r-1} \le \frac{1}{2}u^{-1/2}$ for $u \ge 1$. So these terms contribute at most E_{pow}.

For the zeros, $D\sum_\gamma F\cos = D\sum_\gamma |F| - D\sum_\gamma(|F| - F\cos)$. Every term of the second sum is nonnegative, and for $\pm\gamma$ with $\gamma \in \mathcal{B}$ the two terms together equal $2F(\Delta\gamma)(1-\cos(\gamma\log a))$. Now apply Lemma 16 with $\|F\|_\infty \le \|G\|_1 < 4$. Collecting terms gives (5), with 10^{-12} covering the two tiny errors. □

3.1 The shoulders

Let G be a cosine test of support $1 \le K \le \frac{5}{2}$ and let (a, D) satisfy (4). The sum

$$\Sigma_+ = \sum_{1<|s_p|\le K} \frac{\log p}{\sqrt{p}} G_+(s_p)$$

in (5) runs over the primes on the two *shoulders* $ae^D < p \le ae^{KD}$ and $ae^{-KD} \le p < ae^{-D}$. By (4), $e^{KD} < 1/(1-KD) \le U$; the bound $KD < 10^{-12}$

alone would not give this. For $m \in [ae^{-KD}, ae^{KD}]$ we have $|\log m - 2L| \le KD < 2L(U-1)$ and $e^{-KD/2} \le \sqrt{m/a} \le e^{KD/2}$, so the weight $\theta(m) = \log m/\sqrt{m}$ satisfies

$$U^{-2}\,\frac{2L}{e^L} \le \theta(m) \le U^2\,\frac{2L}{e^L}, \qquad L = \log\sqrt{a}. \tag{6}$$

Lemma 18. *Let $I = (s_1, s_2] \subset (1, K]$ with $s_2 - s_1 = v$. The integers m with $s_m \in I$ form the integers of a real interval of length $t \in [Wv, WvU]$, and those with $-s_m \in I$ form the integers of a real interval of length $t \in [Wv/U, Wv]$; here $W = aD$.*

Proof. The first interval is $(ae^{Ds_1}, ae^{Ds_2}]$, of length $a(e^{Ds_2} - e^{Ds_1})$, which lies between aDv and $aDve^{Ds_2} \le aDvU$ by the mean value theorem. The second is $[ae^{-Ds_2}, ae^{-Ds_1})$, of length between $aDve^{-Ds_2} \ge aDv/U$ and aDv. □

4 Proof of Theorem 4

We cover $x \ge 35 \cdot 10^{12}$ by two ranges: $x \ge 6 \cdot 10^{14}$ (Corollary 11) and $35 \cdot 10^{12} \le x \le 6 \cdot 10^{14}$ (Proposition 22). Throughout, x is real, a, D, P, L, W are as in (1), and we say that x is a *hole* if $(x^\alpha, (x+1)^\alpha) = (ae^{-D}, ae^{D})$ contains no prime. We use repeatedly that

$$P \ge \frac{11}{10}x^{1/10}\frac{x}{x+1}, \qquad \log\frac{1}{D} \le \log\frac{x+1}{1.1}, \qquad 1.1\log x \le L \le 1.1\log(x+1), \tag{7}$$

which follow from $\frac{1}{x+1} \le \log(1+\frac{1}{x}) \le \frac{1}{x}$ and $x^{11/10} \le \sqrt{a} \le (x+1)^{11/10}$. The function $x^{1/10} \cdot x/(x+1)$ is increasing.

4.1 Rigorous arithmetic

The numerical statements of this section, and of Appendices A to F, are checked by the programs listed in Appendix G. They work with exact rational numbers. Transcendental quantities are replaced by rational enclosures, and every rounding goes in the direction that weakens the inequality being checked. The enclosures are the following.

- $\pi \in [\pi_-, \pi_- + 10^{-36}]$, where π_- is the truncation of π to 36 decimals; the program confirms this from Machin's formula with alternating-series remainders.

- $\cos(\pi q)$ for rational q: reduce to $q \in [0, \frac{1}{2}]$ by the symmetries of cosine. The Taylor partial sums of $\cos t$ through degrees 22 and 24 are a lower and an upper bound for $0 \le t \le 2$; they are evaluated at $t = (\pi_- + 10^{-36})q$ and $t = \pi_- q$ respectively, and rounded outwards to multiples of 10^{-18}.
- $\log y$ for rational $y \ge 1$: either the series $\log y' = 2\,\mathrm{artanh}\,\frac{y'-1}{y'+1}$ with an explicit geometric tail bound, after writing $y = 2^k y'$ with $1 \le y' < 2$, rounded outwards to multiples of 2^{-200}; or the correctly rounded decimal logarithm at 48 or 54 significant digits (Python's `decimal` module), widened by more than its rounding error. We write $[\log y]_\pm$ for the two ends.
- $y^{1/10}$ for integer y: the adjacent multiples of 2^{-200} enclosing it, found by integer bisection on tenth powers. We write $[y^{1/10}]_\pm$.

4.2 The twelve-mode test

Let G_{12} be the cosine test (Definition 12) of support $K = 2$ with the twelve weights

$$\begin{aligned} 10^6(w_0, \dots, w_5) &= (462994,\ 232962,\ 17574,\ -31174,\ 472,\ 7016), \\ 10^6(w_6, \dots, w_{11}) &= (-1460,\ -974,\ 377,\ -20,\ 3,\ -11), \end{aligned}$$

and let $F_{12} = \widehat{G_{12}}$. Then $\sum |w_j| = 0.755037 < 1$ and $\|G_{12}\|_1 \le 2K \sum |w_j| < 4$, and by (2)

$$F_{12}(y) = \frac{8}{\pi}\cos(4\pi y) \sum_{j=0}^{11} \frac{(-1)^j k_j w_j}{k_j^2 - 64y^2}, \qquad k_j = 2j + 1.$$

Lemma 19. *For $F = F_{12}$, in the notation of Lemma 16,*

$$F(0) > 1.00000063, \qquad \|F\|_1 < 0.765870, \qquad M_F + 2 \cdot 10^{-6} < -1.37957,$$
$$V_0 < 625, \qquad V_1 < 2000.$$

Proof. Write $\kappa_r = \sum_j k_j^r |w_j|$. By (2), $F(0) = \frac{8}{\pi}\sum_j (-1)^j w_j / k_j$, and the enclosure of π gives $F(0) > 1.00000063$ (the value is $1.0000006340\ldots$).

The integrals on $[0, 16]$. Since $F''(y) = -4\pi^2 \int s^2 G(s) e^{2\pi i y s}\,\mathrm{d}s$, we have $|F''| \le 4\pi^2 \sum |w_j| \int_{-2}^{2} s^2\,\mathrm{d}s = 4\pi^2 \cdot \frac{16}{3} \sum |w_j| < 161$. Let $h = 1/4000$ and $y_i = ih$. The program encloses $F(y_i)$ for $0 \le i \le 64000$ by the closed form, with outward rounding. On each cell $[y_{i-1}, y_i]$ let ℓ_i be the linear interpolant

of F at the two ends. Then $|F-\ell_i| \le \varepsilon := 161h^2/8$ on the cell, and $|\ell_i|$ lies between the smaller and the larger of $|F(y_{i-1})|, |F(y_i)|$ if these have the same sign, and below the larger otherwise. Hence

$$\int_{y_{i-1}}^{y_i} |F| \le \frac{h}{2}\big(|F(y_{i-1})| + |F(y_i)|\big) + \varepsilon h.$$

For the logarithmic moment, on cells with $y_{i-1} \ge 1$, where $\log y \ge 0$, we use $|F| \le \max(|F(y_{i-1})|, |F(y_i)|) + \varepsilon$. On cells with $y_i \le 1$, where $\log y \le 0$, we use $|F| \ge \max\big(0, \min(|F(y_{i-1})|, |F(y_i)|) - \varepsilon\big)$ when the enclosures of $F(y_{i-1})$ and $F(y_i)$ have the same strict sign, and $|F|\log y \le 0$ otherwise. In both cases $\int_{y_{i-1}}^{y_i} \log y \,\mathrm{d}y = y_i \log y_i - y_{i-1}\log y_{i-1} - h$ is evaluated with correctly rounded 48-digit logarithms, rounded in the direction that increases the bound.

Tails. For $y \ge 16$ put $X = 8y \ge 128$ and $\lambda_0 = 1 - (23/128)^2$. Then $X^2 - k_j^2 \ge \lambda_0 X^2$ for all j, as $k_j \le 23$. Writing $\frac{1}{k^2-X^2} = -\frac{1}{X^2} - \frac{k^2}{X^2(X^2-k^2)}$ in the closed form gives

$$|F(y)| \le \frac{c_2}{y^2} + \frac{c_4}{y^4}, \qquad c_2 = \frac{8\,|\sum_j (-1)^j k_j w_j|}{64\pi}, \qquad c_4 = \frac{8\,\kappa_3}{4096\pi\lambda_0}.$$

(The weights are such that $c_2 < 1.2\cdot 10^{-7}$; also $c_4 < 0.0244$.) Hence $\int_{16}^{\infty} |F| \le \frac{c_2}{16} + \frac{c_4}{3\cdot 16^3}$ and $\int_{16}^{\infty} |F(y)|\log y \,\mathrm{d}y \le c_2\frac{\log 16+1}{16} + c_4\frac{\log 16+1/3}{3\cdot 16^3}$. Adding twice the integrals over $[0,16]$ (as F is even) and twice the tails, the program finds $\|F\|_1 < 0.7658691$ and $M_F < -1.3795756$, and checks the stated inequalities exactly.

Variation. By Lemma 13, $|F'| \le 2\pi K^2 \sum |w_j| < 19 < 39$. For $y \ge 16$ write $F(y) = \frac{8}{\pi}\cos(4\pi y) S(y)$ with $S(y) = \sum_j (-1)^j k_j w_j/(k_j^2 - 64y^2)$. Then $|S(y)| \le \kappa_1/(64\lambda_0 y^2)$, and since $\frac{\mathrm{d}}{\mathrm{d}y}(k^2-64y^2)^{-1} = 128y(k^2-64y^2)^{-2}$, also $|S'(y)| \le 2\kappa_1/(64\lambda_0^2 y^3)$. Hence

$$|F'(y)| \le \frac{8}{\pi}\cdot\frac{\kappa_1}{64}\Big(\frac{4\pi}{\lambda_0 y^2} + \frac{2}{\lambda_0^2 y^3}\Big) \le \frac{8}{\pi}\cdot\frac{\kappa_1}{64}\Big(\frac{4\pi}{\lambda_0} + \frac{2}{16\lambda_0^2}\Big)\frac{1}{y^2} < \frac{2}{y^2},$$

as $\kappa_1 = 1.590087$. Therefore $V_0 \le 16\cdot 39 + \int_{16}^{\infty} 2y^{-2}\,\mathrm{d}y < 625$ and $V_1 \le 39\int_1^{16}\log y\,\mathrm{d}y + \int_{16}^{\infty} 2y^{-2}\log y\,\mathrm{d}y = 39(16\log 16 - 15) + \frac{\log 16+1}{8} < 2000$. □

Lemma 20. *Let $h = 1/4000$. There are rational numbers $e_1 \ge e_2 \ge \cdots \ge e_{4000} \ge e_{4001} = 0$ with $e_1 < 0.76$ such that $(G_{12})_+(s) \le e_j$ whenever $1 + (j-1)h < s \le 1 + jh$, for $1 \le j \le 4000$.*

Proof. On $[1,2]$ the function G_{12} is the cosine sum, and $|G_{12}''| \le \sum_j |w_j|(\pi k_j/4)^2 < 16$. For $0 \le i \le 4000$ let $u_i \ge 0$ be the upper bound for $(G_{12})_+(1+ih)$ obtained by replacing each term $w_j \cos(\pi k_j(1+ih)/4)$ by the larger of w_j times the two ends of the enclosure of the cosine, and replacing a negative total by 0. On the cell $[1+(i-1)h, 1+ih]$, G_{12} is at most its linear interpolant plus $16h^2/8 = 2h^2$, so $(G_{12})_+ \le \max(u_{i-1}, u_i) + 2h^2$ there. Put $e_{4001} = 0$ and $e_j = \max\left(e_{j+1}, \max(u_{j-1}, u_j) + 2h^2\right)$ for $j = 4000, \dots, 1$. The program checks $e_1 < 0.76$. □

4.3 The shoulders

We count the primes on the shoulders with the following form of the Brun–Titchmarsh inequality: by [12, Thm. 2], an interval $(y, y+t]$ with $y \ge 0$ and $t > 1$ contains fewer than $2t/\log t$ primes. Hence every real interval of length $t > 1$ contains at most $2t/\log t + 1$ primes.

Let e_j be as in Lemma 20, put $v_j = j/4000$ and $U_1 = 1 + 10^{-10}$, and let $\mathcal{J}$ be the set of j with $e_j > e_{j+1}$.

Lemma 21. *Let $35 \cdot 10^{12} \le x \le 6 \cdot 10^{14}$. Then (a, D) satisfies (4) with $K = 2$, and $P < 34$. Let $P_0, \lambda_-, \lambda_+ > 0$ be such that $P \ge P_0$, $\lambda_- \le L \le \lambda_+$ and $W > 4000U_1$. For $j \in \mathcal{J}$ let $b_j \le \log(P_0 v_j) - \log U_1$ be real numbers with $\lambda_- + b_j > 0$, and put $L_j = \lambda_+$ if $b_j \ge 0$ and $L_j = \lambda_-$ otherwise. Then, for $G = G_{12}$,*

$$\Sigma_+ \le P\mathfrak{b} + 4U^2 e_1 \lambda_+ e^{-L}, \qquad \mathfrak{b} = 8U_1^2 \sum_{j \in \mathcal{J}} (e_j - e_{j+1})\, v_j \, \frac{L_j}{L_j + b_j}.$$

Proof. By (7), $\log a = 2L \ge 2.2 \log(35 \cdot 10^{12}) > 68$. Also $KD = 2.2 \log(1 + 1/x) \le 2.2/x < 10^{-13} < (U-1)/U$, and $P \le 1.1(x+1)^{11/10}/x$, which increases for $x > 10$ and is less than 34 at $x = 6 \cdot 10^{14}$. So (4) holds.

Consider the right shoulder. For $1 + (i-1)/4000 < s \le 1 + i/4000$, Lemma 20 gives

$$(G_{12})_+(s) \le e_i = \sum_{j \ge i} (e_j - e_{j+1}) = \sum_{j \in \mathcal{J}} (e_j - e_{j+1}) \mathbf{1}[s \le 1 + v_j].$$

Let $\theta_{\max} = U^2 \cdot 2L/e^L$, an upper bound for $\theta(p)$ on the shoulders by (6). Then

$$\sum_{1 < s_p \le 2} \theta(p) (G_{12})_+(s_p) \le \theta_{\max} \sum_{j \in \mathcal{J}} (e_j - e_{j+1})\, \pi_j^+,$$

where π_j^+ is the number of primes p with $1 < s_p \le 1 + v_j$. By Lemma 18 these primes lie in a real interval of length $t \in [Wv_j/U, Wv_jU]$. Here $W = aD = Pe^L$, so $t \ge Wv_1/U > 1$ and

$$\log t \ge L + \log(Pv_j) - \log U \ge L + b_j > 0,$$

as $U < U_1$. By the Brun–Titchmarsh inequality above, $\pi_j^+ \le 2Wv_jU/(L + b_j) + 1$. Now $\theta_{\max} \cdot 2Wv_jU = 4U^3Pv_jL$ and $U^3 < U_1^2$. The function $\lambda \mapsto \lambda/(\lambda + b_j)$ is monotone on $[\lambda_-, \lambda_+]$, increasing if $b_j \ge 0$ and decreasing if $b_j < 0$, so $L/(L + b_j) \le L_j/(L_j + b_j)$. Hence the right shoulder contributes at most

$$4U_1^2P \sum_{j \in \mathcal{J}} (e_j - e_{j+1})\, v_j \frac{L_j}{L_j + b_j} + \theta_{\max} \sum_{j \in \mathcal{J}} (e_j - e_{j+1}),$$

and the last sum is e_1. The left shoulder is treated in the same way, with the primes p satisfying $1 < -s_p \le 1 + v_j$, which by Lemma 18 lie in an interval of length in $[Wv_j/U, Wv_j]$. Finally $\theta_{\max} \le U^2 \cdot 2\lambda_+ e^{-L}$. □

4.4 The comparison

For $k \ge 0$ put

$$J_k(x) = 1.1\, k\, x^{1/10} \frac{x}{x+1} - 0.765870 \log \frac{x+1}{1.1} + 1.37957.$$

$l/10^{12}$	$r/10^{12}$	λ_-	λ_+	P_0	$\mathfrak{b} <$	k	$J_k(l) >$	$J_k^\flat >$
35	37	34.2	34.4	24.877	0.098159	0.90184	0.0030	1.4776
37	600	34.2	37.5	25.015	0.098420	0.90150	0.0769	1.4892

Table 1: The two ranges of Proposition 22. The last column is a lower bound for $J_k^\flat = \frac{\mathrm{d}J_k}{\mathrm{d}\log x}$ on $[l, \infty)$. The computed values are $\mathfrak{b} = 0.0981587910\ldots$ and $0.0984189880\ldots$, $J_k(l) \ge 0.0030109\ldots$ and $0.0769651\ldots$.

Proposition 22. *No $x \in [35 \cdot 10^{12}, 6 \cdot 10^{14}]$ is a hole.*

Proof. Let $[l, r]$ be a row of Table 1 and $x \in [l, r]$. The program checks:

(i) $\lambda_- < 1.1[\log l]_-$ and $1.1[\log(r+1)]_+ < \lambda_+$, so $\lambda_- \le L \le \lambda_+$ by (7);

(ii) $P_0 < 1.1[l^{1/10}]_- \, l/(l+1)$, so $P \ge P_0$ by (7), and $P_0\, l\, [l^{1/10}]_- > 4000U_1 \cdot 10^{12}$, which gives $W = Pe^L \ge P_0 x^{11/10} > 4000U_1$;

(iii) with $b_j = [\log(P_0 v_j)]_- - [\log U_1]_+$, that $\lambda_- + b_j > 0$ for $j \in \mathcal{J}$, and that $\mathfrak{b}$, with each term rounded up to a multiple of 10^{-20}, is less than the tabulated value;

(iv) $k < 1.00000063 - \mathfrak{b}$, so $0 < k < F_{12}(0) - \mathfrak{b}$ by Lemma 19;

(v) $1.1\,k\,[l^{1/10}]_- \frac{l}{l+1} - 0.765870\,[\log \frac{l+1}{1.1}]_+ + 1.37957$ exceeds the tabulated lower bound for $J_k(l)$, and $0.11\,k\,[l^{1/10}]_- \frac{l}{l+1} - 0.765870$ exceeds the tabulated lower bound for the derivative.

Suppose that x is a hole. We apply Proposition 17 to G_{12} with $\mathcal{B} = \emptyset$; its hypotheses hold by Lemma 21. We bound the error terms of (5) uniformly on $[35 \cdot 10^{12}, 6 \cdot 10^{14}]$. There $D \le 1.1/l$ with $l = 35 \cdot 10^{12}$, $\log(2\pi/D) \le \log(7(r+1)/1.1)$ with $r = 6 \cdot 10^{14}$, $\ell = \log a + 2D < 2 \cdot 37.5 + 1$, $\sqrt{a} \ge l^{11/10}$ and $P < 34$. With Lemma 19 the program finds

$$E_{\mathrm{pow}} < 7.3 \cdot 10^{-10}, \qquad 2D\Big[\Big(3 + 0.15 \log \frac{2\pi}{D}\Big) V_0 + 0.15 V_1 + 4\Big] < 3.6 \cdot 10^{-10},$$

and $4U^2 e_1 \lambda_+ e^{-L} < 4U^2 \cdot 0.76 \cdot 37.5 / l^{11/10} < 1.5 \cdot 10^{-13}$. Together with the 10^{-12} of (5) the error terms total less than $1.1 \cdot 10^{-9}$, and in particular less than $2 \cdot 10^{-6}$. Hence (5) and Lemma 21 give

$$P\big(F_{12}(0) - \mathfrak{b}\big) \le \|F_{12}\|_1 \log \frac{1}{D} + M_{F_{12}} + 2 \cdot 10^{-6} < 0.765870 \log \frac{1}{D} - 1.37957.$$

By (iv) and (7) the left side is at least $1.1 k x^{1/10} x/(x+1)$, and the right side is at most $0.765870 \log((x+1)/1.1) - 1.37957$. So $J_k(x) < 0$. On the other hand, for $x \ge l$,

$$\begin{aligned} \frac{\mathrm{d}J_k}{\mathrm{d}\log x} &= 1.1\,k\,x^{1/10} \frac{x}{x+1}\Big(\frac{1}{10} + \frac{1}{x+1}\Big) - 0.765870 \frac{x}{x+1} \\ &\ge 0.11\,k\,x^{1/10} \frac{x}{x+1} - 0.765870, \end{aligned}$$

which is increasing in x and positive at $x = l$ by (v). So $J_k(x) \ge J_k(l) > 0$, a contradiction. The two rows cover $[35 \cdot 10^{12}, 6 \cdot 10^{14}]$. □

Proof of Theorem 4. Combine Proposition 22 and Corollary 11. □

Remark. A second, independent program repeats the computations of Lemma 19 and Proposition 22 in ball arithmetic. It evaluates F_{12} as a sum of sinc functions on a different grid ($h = 1/4096$), and gives $J_k(l) > 0.00301095$ and $J_k(l) > 0.07696517$ for the two rows.

5 The computation

In this section we prove Theorem 2 and Proposition 5. A *witness* for a square base m is a prime p with $m^2 < p < (m+1)^2$, and a witness for a power base n is a prime p with $n^{11} < p^5 < (n+1)^{11}$. Both inequalities are checked in exact integer arithmetic. Primality is certified by the following lemma, or, for a few bases, by a Pratt certificate [16].

Lemma 23. *Let $p > 1$ be odd and let F be a power of 2 with $F \mid p-1$, $F^3 > p$ and $3^{(p-1)/2} \equiv -1 \pmod p$. Write $(p-1)/F = sF + t$ with $0 \le t < F$. If $(F+1)^2 > p$, or if $t^2 - 4s$ is negative or not a perfect square, then p is prime.*

Proof. Let q be a prime divisor of p. The order of 3 modulo q divides $p-1$ but not $(p-1)/2$, since $3^{(p-1)/2} \equiv -1 \not\equiv 1 \pmod q$. Hence its 2-adic valuation equals that of $p-1$, so F divides it, and $F \mid q-1$. If p were composite it would have exactly two prime factors counted with multiplicity, as $(F+1)^3 > p$; write $p = (1+aF)(1+bF)$ with $a, b \ge 1$. Then $p \ge (F+1)^2$. If $a+b \ge F$ then $ab \ge a+b-1 \ge F-1$ and $p = 1+(a+b)F+abF^2 \ge 1+F^2+(F-1)F^2 > p$, which is absurd; so $a+b < F$, and $(p-1)/F = abF + (a+b)$ gives $s = ab$, $t = a+b$, and $t^2 - 4s = (a-b)^2$, a perfect square. □

In the computation $F = 2^{\lceil b/3 \rceil}$, where b is the bit length of p, so that $F^3 \ge 2^b > p$; all witnesses are below 2^{100}.

5.1 Retained certificates

Certificates from earlier runs are published at `https://github.com/charlieoneill11/maths-releases`, together with a program that replays them. For squares they cover:

- $70{,}500{,}000{,}000{,}001 \le m \le 70{,}500{,}001{,}100{,}000$, with a witness in each of $(m^2, m(m+1))$ and $(m(m+1), (m+1)^2)$;
- $70{,}500{,}001{,}100{,}001 \le m \le 70{,}500{,}002{,}100{,}000$, and the two pilot ranges $85{,}249{,}999{,}000{,}001 \le m \le 85{,}250{,}000{,}000{,}000$ and $99{,}999{,}999{,}000{,}001 \le m \le 10^{14}$;
- $11{,}894{,}934$ further bases $m > 70{,}500{,}002{,}100{,}000$ outside the two pilot ranges, namely the bases $m = \lfloor \sqrt{p} \rfloor$ for the witnesses p of the retained power certificates for $3{,}883{,}853{,}444{,}451 \le n \le 3{,}883{,}865{,}444{,}451$ ($11{,}894{,}829$ distinct bases), and 105 bases from alternative witnesses at some of the same n.

Since a prime is not a square, every prime p is a witness for the square base $\lfloor\sqrt{p}\rfloor$. For powers, the retained certificates with $n > 5{,}336{,}699{,}231{,}206$ are listed in Table 2.

First n	Last n	Bases
10 000 000 000 000	10 000 000 008 191	8 192
13 672 702 248 661	13 672 702 248 861	201
32 650 000 000 000	32 650 000 002 048	2 049
32 654 400 000 000	32 654 400 004 095	4 096
32 654 570 787 390	32 654 571 787 389	1 000 000
32 654 577 787 390	32 654 578 787 389	1 000 000
32 657 972 050 141	32 657 972 050 141	1
34 999 999 000 000	34 999 999 999 999	1 000 000
		3 014 539

Table 2: Retained power certificates above $n_2 = 5{,}336{,}699{,}231{,}206$. The single base at $32{,}657{,}972{,}050{,}141$ has a Pratt certificate; the others use Lemma 23.

5.2 The 2026 computation

The remaining bases were divided into three workloads:

- *squares*: every m with $70{,}500{,}002{,}100{,}001 \le m \le 10^{14}$ not covered by a retained certificate, $29{,}499{,}984{,}005{,}066$ bases;
- *powers, first band*: every n with $5{,}336{,}699{,}231{,}207 \le n \le 32{,}700{,}000{,}000{,}000$ not in Table 2, $27{,}363{,}298{,}754{,}255$ bases;
- *powers, second band*: every n with $32{,}700{,}000{,}000{,}001 \le n \le 34{,}999{,}998{,}999{,}999$, $2{,}299{,}998{,}999{,}999$ bases.

In total there are $59{,}163{,}281{,}759{,}320$ bases. Of these, $800{,}000{,}000$ were done in pilot runs and the rest in $319{,}488$ production shards ($118{,}030$, $109{,}458$ and $92{,}000$ for the three workloads), listed in a schedule that was fixed before the run; its SHA-256 digest is

`c7a9f9a2c86a11316936f92bba319ad0074437a0708165ae412c11b463758108`.

Witnesses were found by search programs that test candidates $p \equiv 1 \pmod{F}$ in the interval, using vectorised 52-bit modular arithmetic where the size of p allows it (the first power band and most square bases) and GMP otherwise. For a base where the search failed, a slower search produced a

witness with a certificate of Lemma 23 or a Pratt certificate. The correctness of the search programs does not matter, because every witness was then checked by a separate verifier. For each shard, the archive of witnesses was uploaded to storage, fetched back, and compared with its SHA-256 digest. The verifier (a C++ program using GMP, and a Python checker for Pratt certificates) then read the fetched archive and checked:

1. that the bases are exactly the scheduled bases of the shard, each once and in increasing order;
2. the strict inequalities $m^2 < p < (m+1)^2$, or $n^{11} < p^5 < (n+1)^{11}$, for each witness;
3. the primality of each witness, by the hypotheses of Lemma 23 with $F = 2^{\lceil b/3 \rceil}$, or by its Pratt certificate.

A receipt recording the bounds, the digest and the storage key of the shard was written only after the verifier accepted it. The pilot runs worked in the same way and recorded their archives in seven pilot manifests. A final audit reconciled the receipts of all $319{,}488$ shards and the pilot manifests with the schedule.

The computation ran from 22 November 2026, 18:26 PST, until the audit confirmed coverage on 23 November 2026, 21:20 UTC, a total of 18 hours 53 minutes. It used up to 230 machines with 32 logical cores each, about $2{,}924$ machine-hours in all.

Hardware and software. The machines were Blaxel cloud sandboxes provided by Baseten. Each sandbox had 96 GiB of memory and 32 logical CPUs (16 physical cores) of an x86-64 Intel Xeon processor with AVX-512 IFMA, running Linux 6.12. Every sandbox compiled the search programs and the verifier itself, with the system C++ compiler and GMP; the vectorised search uses AVX-512 IFMA when the processor advertises it. The schedule was divided into 230 slots, each owning a fixed slice of the shards. At most 230 sandboxes ran at once. Over the run 317 sandboxes were started, three failed attempts were resumed from their receipts, and all were released. The archives were written to a Hugging Face storage bucket and read back before verification; write volume peaked at about 5.1 TB per hour. From launch to the final audit the fleet verified about 8.7×10^8 witnesses per second. Per active sandbox it verified about 6.7×10^6 per second in the fast power search and in the square search, and 1.2×10^6 per second in the GMP search. The storage bucket of the computation holds $51{,}694{,}242{,}191{,}412$ bytes in $639{,}242$ files; this is the total archived footprint, including the schedule and the receipts as well as the certificate archives. With read access to the bucket,

the audit can be repeated with a published program. In its basic mode the program reads the schedule, the plan file, every production receipt, the seven pilot manifests and the sizes of the stored archives, but no archive contents. It checks that the schedule has the digest above; that each receipt belongs to its shard and lists archives covering the shard's bases in order without gaps; that the pilot manifests have the digests listed in the plan file, list distinct archives, and have the row counts that the plan file assigns to the pilots; that the schedule's shard and row totals agree with the plan file; and that every archive listed in a receipt or a pilot manifest is present with its recorded size. In replay mode it also fetches the archive of every production shard, compares it with the digest in the receipt, and runs the verifier on it with the shard's bounds and list of excluded bases; this repeats the arithmetic for all $59{,}162{,}481{,}759{,}320$ production bases, and the work can be split into independent slots. The published digest authenticates the schedule but not the plan file; the plan file's totals and its digests of the seven pilot manifests should be compared with the copy published with the programs. For the $800{,}000{,}000$ pilot bases the program does not read the archives, so a pilot archive altered without a change of length would pass. Each pilot archive was fetched back and verified when it was made, before its manifest entry was written; that check is not repeated by the released programs. The program also does not re-derive from the ranges themselves that the production shards, the pilot ranges, the excluded bases and the retained certificates cover each workload exactly once. That was checked by the planning program when the schedule was made, from the pilot manifests and the retained archives.

5.3 Status of the evidence

Three kinds of evidence support Theorem 2 and Proposition 5.

Public certificates. The replay program for the published certificates reads only published files. It rebuilds the decoder and the GMP verifier, verifies every retained witness and Pratt certificate, and checks the coverage counts of Section 5; it needs no network access. With [20] and Proposition 3, the retained certificates prove Theorem 1 for every $n \le 3{,}883{,}865{,}444{,}451$ and Theorem 2 for every $m \le 70{,}500{,}002{,}100{,}000$. They also cover isolated ranges: for squares, the two pilot ranges and the $11{,}894{,}934$ further bases listed in Section 5; for powers, the rows of Table 2 and, through the square pilot ranges, the bases $4{,}616{,}019{,}348{,}273 \le n \le 4{,}616{,}019{,}397{,}498$ and $5{,}336{,}699{,}182{,}690 \le n \le 5{,}336{,}699{,}231{,}206$. The analytic results cover

every real $x \geq 35 \cdot 10^{12}$ (Theorem 4), and every real $x \geq 32{,}746{,}701{,}927{,}355$ by Theorem 6(a).

The original computation. For every other base, $3{,}883{,}865{,}444{,}451 < n < 35 \cdot 10^{12}$ and $70{,}500{,}002{,}100{,}000 < m \leq 10^{14}$, Proposition 5 and Theorem 2 rest on the 2026 computation: on the verification of each archive after it had been stored and fetched back, on the receipts and pilot manifests written after that verification, and on the final audit. Its programs, the schedule digest, the pilot-manifest digests and the audited totals of all 230 slots are public; its archive is not.

Independent reproduction. A referee with read access to the bucket replayed the 96 archives listed in three of the seven pilot manifests, 160,000,000 rows (64,000,000 square, 64,000,000 fast-power and 32,000,000 GMP bases); the archives matched the manifests. The other 640,000,000 pilot rows and the 59,162,481,759,320 production rows have not been replayed by anyone other than the original run.

A full replay of the pilot is not implemented. It would authenticate the plan file against the published record and the seven pilot manifests against their digests there; fetch every pilot archive and compare it with its digest; decode the witnesses according to their recorded kind and range and verify primality and the strict inequalities; and reconcile the pilot ranges, the production shards, the excluded bases and the retained certificates without gaps or double counting. It needs read access to the bucket. Replaying the production shards needs the same access and the replay mode of the audit program.

In short, Theorems 1 and 2 are proved by the computation described here, but they have not been verified independently in full. An independent check is available for the public prefixes and isolated ranges above and for 1.6×10^8 further bases; the remaining 5.9×10^{13} checks rest on the original run.

5.4 Proofs

Proof of Theorem 2. For $m \leq 7.05 \times 10^{13}$ this is the computation of Sorenson and Webster [20] (for $m = 1$, the interval $(1, 4)$ contains 2). The remaining $10^{14} - 70{,}500{,}000{,}000{,}000 = 29{,}500{,}000{,}000{,}000$ bases are covered by the retained certificates ($1{,}100{,}000 + 1{,}000{,}000 + 2{,}000{,}000 + 11{,}894{,}934$ bases) and the square workload (29,499,984,005,066 bases). □

Proof of Proposition 5. Let $n_2 = 5{,}336{,}699{,}231{,}206$, the largest integer with $n_2^{11} \leq 10^{140}$. For $n_1 < n \leq n_2$ we have $n > 400$, so Lemma 9 gives a square

base $m \le 10^{14}$ with $(m^2, (m+1)^2) \subset (n^\alpha, (n+1)^\alpha)$, and Theorem 2 gives a prime there. The 29,663,300,768,793 bases $n_2 < n < 35 \cdot 10^{12}$ are covered by Table 2 (3,014,539 bases) and the two power workloads (27,363,298,754,255 + 2,299,998,999,999 bases). □

A A sharper cutoff

In this appendix and Appendices B to F we prove Theorem 6. None of this is used in the proof of Theorem 1. By Theorem 4 it suffices to show that no x between the stated cutoff and $35 \cdot 10^{12}$ is a hole. We show this up to $3.9 \cdot 10^{13}$, which is the range the program checks: Proposition 42 gives part (b) and Proposition 43 gives part (a).

The argument has the same structure as Section 4: Proposition 17, a bound for the shoulder sum Σ_+, and a comparison function. It uses a test function with twenty cosine modes and support $\frac{5}{2}$ (Appendix E), and it bounds Σ_+ more sharply in three ways.

1. A weighted large sieve over the Farey fractions with squarefree denominators replaces the Brun–Titchmarsh inequality. Discarding the other fractions increases the spacing of the remaining ones, and the spacing enters the Montgomery–Vaughan weights (Appendix C).
2. The sieve counts every integer free of small prime factors, including the products pq of two large primes. A lower bound for the number of such semiprimes on the shoulders can therefore be subtracted. We obtain it from the distribution of primes at the much smaller heights m/q, using the complete histogram of prime gaps below 4×10^{18} [14] through a convexity inequality (Lemma 34), and, in part (b) only, the table of maximal prime gaps [19] through a wheel argument (Lemma 35).
3. In part (b), on a short final range, we retain the signed contribution of the first 1.4×10^6 zeros [15, 10].

The shoulder sum is first decomposed into layers (Appendix B), so that the sieve can be applied to intervals.

B Layers on the shoulders

In this section G is a cosine test of support $1 \le K \le \frac{5}{2}$, (a, D) satisfy (4), and Σ_+ is the shoulder sum of Section 3.1. Fix an integer $M \ge 1$ such that $(K-1)M$ is an integer, and cut $[1, K]$ into cells $C_i = (1 + \frac{i-1}{M}, 1 + \frac{i}{M}]$,

$1 \le i \le n = (K-1)M$. Let $g = \sum_i h_i \mathbf{1}_{C_i}$ with $h_i \ge 0$ and $g(s) \ge G_+(s)$ for $1 < s \le K$. For each $\lambda \ge 0$ the set $\{g > \lambda\}$ is a disjoint union of maximal runs of consecutive cells. As λ increases, these runs change only at the finitely many values h_i. Hence there are finitely many *strips* (I_κ, η_κ), where I_κ is a run of w_κ consecutive cells and $\eta_\kappa > 0$, such that

$$g = \sum_\kappa \eta_\kappa \mathbf{1}_{I_\kappa}, \qquad \sum_\kappa \eta_\kappa = \int_0^\infty \#\{\text{runs of } \{g > \lambda\}\}\, \mathrm{d}\lambda. \tag{8}$$

(A stack algorithm produces them in one pass over the cells.) Write $v_\kappa = w_\kappa / M$ for the log-width of I_κ.

Lemma 24. *For each strip κ and each sign, let $J_\kappa^\pm$ be the set of integers m with $\pm s_m \in I_\kappa$, and let $\mathcal{A}_\kappa^\pm \subset J_\kappa^\pm$ contain every prime of $J_\kappa^\pm$. Let $\mathcal{S}$ be a set of composite integers m with $1 < |s_m| \le K$ such that $m \in \mathcal{A}_\kappa^\pm$ whenever $m \in J_\kappa^\pm$. Let $0 \le g_- \le G_+$ on $(1, K]$. Then*

$$\Sigma_+ \le U^2 \frac{2L}{e^L} \Big[\sum_\kappa \eta_\kappa \big(|\mathcal{A}_\kappa^+| + |\mathcal{A}_\kappa^-| \big) - \sum_{m \in \mathcal{S}} g_-(|s_m|) \Big].$$

Proof. By (6), $G_+ \le g$ and (8),

$$\Sigma_+ \le U^2 \frac{2L}{e^L} \sum_{1 < |s_p| \le K} g(|s_p|) = U^2 \frac{2L}{e^L} \sum_\kappa \eta_\kappa \big(\pi(J_\kappa^+) + \pi(J_\kappa^-) \big),$$

where $\pi(J)$ is the number of primes in J. Since $\mathcal{A}_\kappa^\pm$ contains the primes of $J_\kappa^\pm$ and the elements of $\mathcal{S} \cap J_\kappa^\pm$, which are composite, $\pi(J_\kappa^\pm) \le |\mathcal{A}_\kappa^\pm| - |\mathcal{S} \cap J_\kappa^\pm|$. Finally $\sum_\kappa \eta_\kappa (|\mathcal{S} \cap J_\kappa^+| + |\mathcal{S} \cap J_\kappa^-|) = \sum_{m \in \mathcal{S}} g(|s_m|) \ge \sum_{m \in \mathcal{S}} g_-(|s_m|)$. □

In Appendix C we take for $\mathcal{A}_\kappa^\pm$ the elements of $J_\kappa^\pm$ free of prime factors up to a radius $z_\kappa < 2 \cdot 10^8$, and bound $|\mathcal{A}_\kappa^\pm|$ by a large sieve. In Appendix D we take for $\mathcal{S}$ products of two primes larger than $2 \cdot 10^8$; these lie in every $\mathcal{A}_\kappa^\pm$ that contains them, as the lemma requires.

C A weighted sieve over squarefree Farey fractions

C.1 The weighted large sieve

Let $\rho = \sqrt{1 + \frac{2}{3}\sqrt{6/5}} = 1.31540\ldots$ be Preissmann's constant, and put $\mathcal{K} = 1.316 > \rho$.

Lemma 25. *Let $R \geq 2$, let $x_1, \dots, x_R$ be distinct points of $\mathbb{R}/\mathbb{Z}$ and $\delta_r = \min_{s \neq r} \|x_r - x_s\| > 0$. Let M and $N \geq 1$ be integers, let $(c_n)_{M < n \leq M+N}$ be complex numbers and $S(x) = \sum_n c_n e(nx)$. Then*

$$\sum_{r=1}^{R} \frac{|S(x_r)|^2}{N + \mathcal{K}\delta_r^{-1}} \leq \sum_n |c_n|^2.$$

Proof. Preissmann [17] proved the weighted Hilbert inequality

$$\Big| \sum_{r \neq s} \frac{u_r \bar{u}_s}{\lambda_r - \lambda_s} \Big| \leq \rho\pi \sum_r \frac{|u_r|^2}{\delta'_r} \tag{9}$$

for distinct reals λ_r with $\delta'_r = \min_{s \neq r} |\lambda_r - \lambda_s|$, improving the constant $\frac{3}{2}$ in the real-line form of Montgomery and Vaughan [13, Theorem 2, (1.7)]. We first deduce the periodic form

$$\Big| \sum_{r \neq s} u_r \bar{u}_s \csc \pi(x_r - x_s) \Big| \leq \rho \sum_r \frac{|u_r|^2}{\delta_r}. \tag{10}$$

Represent each x_r by a real number in $[0, 1)$. For an integer $Q \geq 1$ apply (9) to the points $\lambda_{r,j} = x_r + j$, $|j| \leq Q$, with coefficients $(-1)^j u_r$; their minimal distances are at least δ_r. Divide by $2Q+1$ and let $Q \to \infty$. The pairs $(r, j), (r, j')$ with $j \neq j'$ contribute $\sum_r |u_r|^2 \sum_{0<|k|\leq 2Q} (-1)^k (1 - \frac{|k|}{2Q+1})/k = 0$. For $r \neq s$ the pairs contribute $u_r \bar{u}_s$ times the Cesàro means of the symmetric partial sums of $\sum_k (-1)^k/(x_r - x_s + k)$, which converge to $\pi \csc \pi(x_r - x_s)$. This gives (10).

Now put $\omega_r = (N + \mathcal{K}/\delta_r)^{-1}$ and consider the operator $T : (c_n) \mapsto (\omega_r^{1/2} S(x_r))_r$. For complex (b_r),

$$\|T^* b\|^2 = \sum_n \Big| \sum_r b_r \omega_r^{1/2} e(-n x_r) \Big|^2 = \sum_{r,s} b_r \bar{b}_s (\omega_r \omega_s)^{1/2} \sum_n e\big(n(x_s - x_r)\big).$$

The diagonal terms give $N \sum_r |b_r|^2 \omega_r$. For $\theta \notin \mathbb{Z}$, $\sum_{M<n\leq M+N} e(n\theta) = e(\beta\theta) \sin(\pi N\theta)/\sin(\pi\theta)$ with $\beta = M + \frac{N+1}{2}$, and $\sin(\pi N\theta) = \frac{1}{2i}(e(\frac{N}{2}\theta) - e(-\frac{N}{2}\theta))$. Each of the two resulting off-diagonal forms is of the shape (10) with $u_r = b_r \omega_r^{1/2} e(-(\beta \pm \frac{N}{2}) x_r)$, so the off-diagonal terms have absolute value at most $\rho \sum_r |b_r|^2 \omega_r/\delta_r$. Hence $\|T^* b\|^2 \leq \sum_r |b_r|^2 \omega_r (N + \rho/\delta_r) \leq \sum_r |b_r|^2$, so $\|T\| = \|T^*\| \leq 1$. □

C.2 The sifted count

Fix $\lambda = 0.83$ and $T = \mathcal{K}\lambda = 1.09228$. Let J be a real interval containing exactly N integers, where $\lambda N \geq 4$, let $z = \sqrt{\lambda N}$ and $B = \lfloor z \rfloor \geq 2$, and let $\mathcal{A}$ be the set of integers of J with no prime factor $\leq z$. Let $\mathcal{R}$ be the set of fractions a/q with $1 \leq q \leq B$, q squarefree, $0 \leq a < q$ and $(a, q) = 1$, viewed as points of $\mathbb{R}/\mathbb{Z}$; as $B \geq 2$, $\mathcal{R}$ contains at least the two points 0 and $\frac{1}{2}$. For $a/q \in \mathcal{R}$ let $\delta_{a,q}$ be the distance from a/q to the nearest other point of $\mathcal{R}$.

Lemma 26. *$|\mathcal{A}| \leq N/H$, where*

$$H = \sum_{q \leq z} \frac{\mu^2(q)}{\varphi(q)} \Big(1 + \frac{\mathcal{K}}{N\varphi(q)} \sum_{a \bmod q}^{*} \delta_{a,q}^{-1} \Big)^{-1}.$$

Proof. Put $Z = |\mathcal{A}|$ and $S(x) = \sum_{n \in \mathcal{A}} e(nx)$. For squarefree $q \leq z$ every $n \in \mathcal{A}$ is coprime to q, so $\sum_a^* S(a/q) = \sum_{n \in \mathcal{A}} c_q(n) = \mu(q) Z$, where c_q is Ramanujan's sum. By Cauchy–Schwarz,

$$\mu^2(q) Z^2 \leq \sum_a^* \frac{|S(a/q)|^2}{N + \mathcal{K}\delta_{a,q}^{-1}} \cdot \sum_a^* \big(N + \mathcal{K}\delta_{a,q}^{-1} \big).$$

Divide by the last factor, sum over $q \leq z$ and apply Lemma 25 to the points of $\mathcal{R}$ and $c_n = \mathbf{1}_{\mathcal{A}}(n)$. This gives $Z^2 \sum_q \mu^2(q) / \sum_a^* (N + \mathcal{K}\delta_{a,q}^{-1}) \leq Z$, which is $Z \leq N/H$. □

C.3 Farey spacing

For squarefree $1 < q \leq B$ and $(a, q) = 1$, let b_- and b_+ be the integers in $(B - q, B]$ with $ab_- \equiv 1$ and $ab_+ \equiv -1 \pmod q$. They are the denominators of the neighbours of a/q in the Farey sequence of order B, at distances $1/(qb_-)$ and $1/(qb_+)$ [8, Ch. III]. Every point of $\mathcal{R}$ belongs to that Farey sequence, so

$$\delta_{a,q}^{-1} \leq q\, b_{\mathrm{hi}}, \qquad b_{\mathrm{hi}} = \max(b_-, b_+), \quad b_{\mathrm{lo}} = \min(b_-, b_+). \tag{11}$$

Put $r = (B - b_-)/q$ and $r' = (B - b_+)/q$, both in $[0, 1)$, so $b_{\mathrm{hi}} = B - q\min(r, r')$. Since $b_- + b_+ \equiv 0 \pmod q$ we have $r' = \{v_0 - r\}$ with $v_0 = \{2B/q\}$. As a runs over the reduced residues modulo q, b_- runs over the integers of $(B - q, B]$ coprime to q, and so does b_+. Write $\omega(q)$ for the number of prime factors of q and $e_q = 2^{\omega(q)}/\varphi(q)$. By inclusion–exclusion over the divisors of q, for every interval $I \subset [0, 1)$

$$\Big| \#\{a : r \in I\} - \varphi(q)|I| \Big| \leq 2^{\omega(q)}, \tag{12}$$

and the same holds for r'. Put $A(v) = \frac{1}{4} + \frac{v(1-v)}{2}$ for $v \in [0,1)$; extended periodically, A is continuous and $\frac{1}{2}$-Lipschitz on $\mathbb{R}/\mathbb{Z}$.

Lemma 27. *For squarefree $1 < q \le B$,*

$$\frac{1}{\varphi(q)} \sum_a{}^{*} \min(r, r') \ge A(v_0) - 2e_q \ge A(v) - 2e_q - \frac{1}{q}, \qquad v = \{2z/q\}.$$

Proof. Let $h(t) = \min(t, \{v_0 - t\})$ on $[0,1)$. On $[0, v_0]$ it is $\min(t, v_0 - t)$ and on $(v_0, 1)$ it is $\min(t, 1 + v_0 - t)$, so each superlevel set $\{h > \mu\}$ is a union of at most two intervals and $0 \le h < 1$. By (12),

$$\begin{aligned} \frac{1}{\varphi(q)} \sum_a{}^{*} h(r) &= \int_0^1 \frac{\#\{a : h(r) > \mu\}}{\varphi(q)} \, \mathrm{d}\mu \\ &\ge \int_0^1 |\{h > \mu\}| \, \mathrm{d}\mu - 2e_q = \int_0^1 h - 2e_q. \end{aligned}$$

Direct integration gives $\int_0^{v_0} \min(t, v_0 - t)\,\mathrm{d}t = v_0^2/4$ and $\int_{v_0}^1 \min(t, 1 + v_0 - t)\,\mathrm{d}t = \frac{1}{4}(1+v_0)^2 - v_0^2$, with sum $A(v_0)$. Finally $\|2B/q - 2z/q\| < 2/q$ in $\mathbb{R}/\mathbb{Z}$, so $A(v_0) \ge A(v) - 1/q$. □

The second improvement comes from the fractions we discarded. Let $\mathcal{P} = \{2, 3, 5, 7\}$ and, for squarefree q,

$$c_q = 1 - \prod_{p \in \mathcal{P},\, p \nmid q} \Big(1 - \frac{1}{p^2}\Big).$$

Lemma 28. *For squarefree $1 < q \le B$, with $s = q/z$,*

$$\frac{1}{\varphi(q)} \sum_a{}^{*} \delta_{a,q}^{-1} \le q\Big[z - q\Big(A(v) + \frac{c_q}{4(1+s)} - 62e_q - \frac{2}{q}\Big)\Big].$$

Without the term $c_q/(4(1+s))$ the bound holds with $62e_q + 2/q$ replaced by $2e_q + 1/q$.

Proof. The second statement follows from (11), $b_{\mathrm{hi}} = B - q\min(r, r') \le z - q\min(r, r')$ and Lemma 27.

Suppose that $p^2 \mid b_{\mathrm{hi}}$ for some $p \in \mathcal{P}$ with $p \nmid q$. Then the Farey neighbour with denominator b_{hi} is not in $\mathcal{R}$. On that side the next Farey fraction lies at a further distance at least $1/(b_{\mathrm{hi}}B) \ge 1/(b_{\mathrm{hi}}z)$, so every point of $\mathcal{R}$ on that

side is at distance at least $1/(qb_{\mathrm{hi}}) + 1/(zb_{\mathrm{hi}})$ from a/q. On the other side the distance is at least $1/(qb_{\mathrm{lo}})$. Hence

$$\delta_{a,q}^{-1} \le \max\Big(qb_{\mathrm{lo}}, \frac{qzb_{\mathrm{hi}}}{z+q}\Big) = qb_{\mathrm{hi}} - q\min\Big(b_{\mathrm{hi}} - b_{\mathrm{lo}}, \frac{qb_{\mathrm{hi}}}{z+q}\Big).$$

Here $b_{\mathrm{hi}} - b_{\mathrm{lo}} = q|r - r'|$, and since $b_{\mathrm{hi}} \ge z - 1 - q\min(r, r')$ and $1 - s\min(r, r') \ge 1 - \min(r, r') \ge |r - r'|$,

$$\frac{b_{\mathrm{hi}}}{z+q} \ge \frac{1 - 1/z - s\min(r, r')}{1+s} \ge \frac{|r - r'| - 1/z}{1+s}.$$

So the saving is at least $\frac{q^2}{1+s}(|r - r'| - \frac{1}{z})$. It remains to show

$$\frac{1}{\varphi(q)} \sum_{a:\, \exists p \in \mathcal{P},\, p \nmid q,\, p^2 \mid b_{\mathrm{hi}}}^{*} |r - r'| \ge \frac{c_q}{4} - 60e_q. \tag{13}$$

Given this, the total saving divided by $\varphi(q)$ is at least $\frac{q^2}{1+s}(\frac{c_q}{4} - 60e_q - \frac{1}{z}) \ge q^2(\frac{c_q}{4(1+s)} - 60e_q - \frac{1}{q})$, and combining with Lemma 27 proves the lemma.

For (13), write the left side as $\int_0^1 \varphi(q)^{-1} \#\{a : \text{condition}, \, |r - r'| > \mu\}\, \mathrm{d}\mu$ and split according to whether $r < r'$ (then $b_{\mathrm{hi}} = b_-$) or $r > r'$ (then $b_{\mathrm{hi}} = b_+$). In the first case the condition $r' - r > \mu$ defines a union of at most two intervals of r, that is of at most two intervals of b_-. The number of integers b in an interval of length ℓ that are coprime to q and divisible by p^2 for some $p \in \mathcal{P}$ with $p \nmid q$ is, by inclusion–exclusion over the at most 15 nonempty subsets of $\mathcal{P}$ and over the divisors of q, equal to $c_q\varphi(q)\ell/q$ up to an error of at most $15 \cdot 2^{\omega(q)}$. The second case is symmetric, with r' in place of r. So, uniformly in μ, the count is at least $\varphi(q)c_q|\{t : |t - \{v_0 - t\}| > \mu\}| - 60 \cdot 2^{\omega(q)}$. Integrating over μ gives at least $c_q \int_0^1 |t - \{v_0 - t\}|\, \mathrm{d}t - 60e_q = c_q\frac{v_0^2 + (1-v_0)^2}{2} - 60e_q \ge \frac{c_q}{4} - 60e_q$. □

C.4 A lower bound for H

For $0 < s \le 1$ put $v(s) = \{2/s\}$ and

$$w(s) = \frac{1}{1 + Ts\big(1 - sA(v(s))\big)}, \qquad f(s) = \frac{Ts}{4(1+s)(1+Ts)^2}.$$

Proposition 29. *Suppose $z \le 2 \cdot 10^8$. Then*

$$H \ge \sum_{q \le z} \frac{\mu^2(q)}{\varphi(q)} w\Big(\frac{q}{z}\Big) + \frac{1}{z}\sum_{q \le z} \mu^2(q) c_q f\Big(\frac{q}{z}\Big) - \frac{16250\,T}{z},$$

and also $H \ge \sum_{q \le z} \mu^2(q)\varphi(q)^{-1} w(q/z) - 530T/z$.

Proof. Every $q \le 2\cdot 10^8$ has at most eight prime factors, as $2\cdot 3\cdots 23 > 2\cdot 10^8$, and so $q/\varphi(q) \le \prod_{p\le 19}\frac{p}{p-1} < 6$.

For $q = 1$ the nearest point to 0 in $\mathcal{R}$ is at distance at least $1/B$, so the weight is at least $(1+\mathcal{K}z/N)^{-1} = w(1/z)$-type bounds hold trivially; precisely, $\delta_{0,1}^{-1} \le B \le z$ and the bound of Lemma 28 with $q=1$ is weaker than this. For squarefree $1 < q \le z$ write $s = q/z$ and $\alpha_q = A(v) + \frac{c_q}{4(1+s)} - 62e_q - \frac{2}{q}$. By Lemma 28, and since $\mathcal{K}z^2/N = T$, the weight of q in H is at least

$$W(s,\alpha_q), \qquad W(s,\alpha) = \frac{1}{1+Ts(1-s\alpha)}.$$

The function $\alpha \mapsto W(s,\alpha)$ is increasing for $\alpha < 1/s$, with derivative $Ts^2/(1+Ts(1-s\alpha))^2$. This derivative is at most Ts^2 for $\alpha \le 1/s$, and at least $Ts^2/(1+Ts)^2$ for $0 \le \alpha \le 1$. Since $0 \le A(v) \le A(v) + \frac{c_q}{4(1+s)} \le 1$,

$$W(s,\alpha_q) \ge w(s) + \frac{c_q}{4(1+s)}\frac{Ts^2}{(1+Ts)^2} - \Big(62e_q + \frac{2}{q}\Big)Ts^2.$$

Multiply by $\mu^2(q)/\varphi(q)$ and sum. In the middle term use $1/\varphi(q) \ge 1/q$, which gives $\frac{1}{z}\sum \mu^2(q)c_q f(q/z)$. The error term is

$$\frac{T}{z^2}\sum_{q\le z}\mu^2(q)\Big(62\cdot 2^{\omega(q)}\frac{q^2}{\varphi(q)^2} + 2\frac{q}{\varphi(q)}\Big) \le \frac{T}{z^2}\big(62\cdot 13z(1+\log z) + 12z\big) \le \frac{16250T}{z}.$$

Here we used $q/\varphi(q) < 6$ and

$$\sum_{q\le z}\mu^2(q)2^{\omega(q)}\Big(\frac{q}{\varphi(q)}\Big)^2 \le 13z(1+\log z).$$

To see the last bound, note that on squarefree q the function $2^{\omega(q)}(q/\varphi(q))^2$ equals $(\tau * g)(q)$, where g is multiplicative, supported on squarefree integers, with $g(p) = 2((1-1/p)^{-2}-1) \ge 0$. Hence the sum is at most $\sum_{d\le z} g(d)\sum_{e\le z/d}\tau(e) \le z(1+\log z)\sum_d g(d)/d$, and $\sum_d g(d)/d = \prod_p(1+g(p)/p) < 13$: the factors with $p \le 47$ are computed exactly, and $\log$ of the remaining product is at most $\sum_{p>47} g(p)/p \le 5\sum_{n>47} n^{-2} \le 5/47$. Finally $62\cdot 13(1+\log(2\cdot 10^8)) + 12 < 16250$. The second statement is proved in the same way from the second part of Lemma 28: the error is at most $\frac{T}{z^2}(2\cdot 13z(1+\log z) + 6z) \le 530T/z$. □

Lemma 30. *For $t \ge 1$,*

$$\Big|\sum_{q\le t}\frac{\mu^2(q)}{\varphi(q)} - \log t - \eta_1\Big| \le \frac{2.44}{\sqrt{t}}, \qquad \eta_1 = \gamma_0 + \sum_p \frac{\log p}{p(p-1)} > \eta := 1.33252,$$

where γ_0 is Euler's constant.

Proof. The inequality is due to Ramaré [18]. For the lower bound on η_1 we use $\gamma_0 > H_{500} - \log 500 - \frac{1}{1000}$ and the sum over $p \le 20000$, evaluated with outward rounding. □

Lemma 31. *Let $J_w = \int_0^1 (w(s)-1)\frac{\mathrm{d}s}{s}$. Then*

$$C_0 := \log\lambda + 2\eta + 2J_w > 1.1452, \qquad w(1) + \int_0^1 |w'(s)|\frac{\mathrm{d}s}{\sqrt{s}} < 2.287,$$
$$\frac{1}{z}\sum_{q\le z}\mu^2(q)c_q f\Big(\frac{q}{z}\Big) \ge 0.00491 - 0.11\Big(\frac{2}{\sqrt{z}} + \frac{2}{z}\Big).$$

Proof. On a cell $[s_0, s_1]$ let $\underline{A}$ be a lower bound for $A(v(s))$: if $2/s$ crosses no integer on the cell, $A(v)$ is at least its smaller endpoint value, as $v(1-v)$ is concave; otherwise $\underline{A} = \frac{1}{4}$. The integrand $(w-1)/s = -u/(1+su)$ with $u = T(1 - sA(v(s)))$ is decreasing in u and increasing in s, so on the cell it is at least $-u_0/(1+s_0u_0)$ with $u_0 = T(1 - s_0\underline{A})$. Summing these lower values over 32768 equal cells gives $C_0 > 1.1452$.

Away from the points where $2/s$ is an integer, $w'(s) = -2T(1-v-sA(v))/(1+Ts(1-sA(v)))^2$, because $\frac{d}{ds}(s - s^2A(\{2/s\})) = 2(1-v-sA(v))$. Substitute $s = t^2$, so the integral is $2\int_0^1 |w'(t^2)|\,\mathrm{d}t$, and on 2048 equal cells in t bound $|1-v-sA|$ and the denominator using $\frac{1}{4} \le A \le \frac{3}{8}$ and the endpoint values of v. With $w(1) = 1/(1+3T/4)$ the total is below 2.287.

For the last bound, $c_q \ge \sum_{p\in\mathcal{P}, p\nmid q} c_p$ with $c_p = p^{-2}\prod_{r\in\mathcal{P}, r<p}(1-r^{-2})$, since the union of the events $p^2 \mid b$ has density $\sum_p p^{-2}\prod_{r<p}(1-r^{-2})$ over the primes of $\mathcal{P}$ not dividing q. Fix $p \in \mathcal{P}$. In a real interval of length $\ell \subset [0,z]$ the number of squarefree integers not divisible by p is at least $\rho_p\ell - 2\sqrt{z} - 2$, where $\rho_p = (1-\frac{1}{p})(1-\sum_r r^{-2} + p^{-2})$, the sum running over all primes: count the integers prime to p, then remove for each prime $r \ne p$ with $r^2 \le z$ those divisible by r^2, each class costing at most 2 beyond its density. We used $\sum_r r^{-2} < 0.462$, from the primes up to 101 and the tail $\sum_{n>101} n^{-2} < 1/101$. The function f is nonnegative and unimodal on $[0,1]$,

as f' has the sign of $1-Ts-2Ts^2$, so its superlevel sets are intervals, and $f \le T/4$. Integrating the count over the levels of f gives

$$\frac{1}{z}\sum_{q\le z}\mu^2(q)c_q f\Big(\frac{q}{z}\Big) \ge \sum_{p\in\mathcal{P}} c_p\Big(\rho_p\int_0^1 f - \frac{T}{4}\Big(\frac{2}{\sqrt{z}}+\frac{2}{z}\Big)\Big).$$

Lower rectangles on 4096 cells give $\sum_p c_p\rho_p\int_0^1 f > 0.00491$, and $\frac{T}{4}\sum_p c_p < 0.11$. $\square$

Proposition 32. *Let J be a real interval of length $t \ge 10^{10}$ containing N integers, with $\sqrt{\lambda N} \le 2\cdot 10^8$, and let z_0 be an integer with $10^4 \le z_0 \le \sqrt{\lambda N}$. Put $c(z_0)=\max\{c_1(z_0),c_2(z_0)\}$, where $\vartheta = 2.44\cdot 2.287$ and*

$$c_1(z_0) = C_0 - 2\Big(\frac{\vartheta}{\sqrt{z_0}} + \frac{\eta T}{z_0} + \frac{530T}{z_0}\Big),$$
$$c_2(z_0) = C_0 + 2\Big(0.00491 - 0.11\Big(\frac{2}{\sqrt{z_0}}+\frac{2}{z_0}\Big) - \frac{\vartheta}{\sqrt{z_0}} - \frac{\eta T}{z_0} - \frac{16250T}{z_0}\Big).$$

Then the number of integers in J with no prime factor at most $\sqrt{\lambda N}$ is at most $2t/(\log t + c(z_0))+1$.

Proof. Put $z=\sqrt{\lambda N}$. By partial summation with Lemma 30, writing $\Sigma(u)=\sum_{q\le u}\mu^2(q)/\varphi(q)$,

$$\sum_{q\le z}\frac{\mu^2(q)}{\varphi(q)}w\Big(\frac{q}{z}\Big) = w(1)\Sigma(z) - \int_1^z \Sigma(u)\frac{d}{du}w\Big(\frac{u}{z}\Big)\,\mathrm{d}u.$$

The main terms $\log u+\eta_1$ give $\int_{1/z}^1 w(s)\frac{\mathrm{d}s}{s} + \eta_1 w(1/z) \ge \log z + J_w + \eta(1-T/z)$, using $w\le 1$ and $w(1/z)\ge 1-T/z$. The error terms are at most $\frac{2.44}{\sqrt{z}}(w(1)+\int_0^1 |w'(s)|s^{-1/2}\,\mathrm{d}s) < \frac{\vartheta}{\sqrt{z}}$ by Lemma 31. With Proposition 29 and $2\log z = \log N + \log\lambda$ this gives $2H \ge \log N + c(z)$, and $c(z)\ge c(z_0)$ since c is increasing. As $z_0\ge 10^4$, $c(z_0)\ge c_1(10^4) > C_0 - 2(0.0559+0.0002+0.058) > 0.9$, so $\log N + c(z_0) > 1$ and the bounds below are positive. (Without a lower bound on z_0 the statement fails: for $z_0=1$ and $t=N=10^{10}$ one has $c(1)<-1169$, and $2t/(\log t + c(1))+1<0$.) By Lemma 26 the count is at most $N/H \le 2N/(\log N + c(z_0))$. As $N\le t+1$ and $u\mapsto 2u/(\log u + c)$ is increasing for $\log u + c > 1$, this is at most $2(t+1)/(\log(t+1)+c(z_0)) \le 2t/(\log t + c(z_0))+1$. $\square$

Remark. The expression $c_1(z_0)$ uses only the mean spacing of Lemma 27; $c_2(z_0)$ also uses the saving from the discarded Farey neighbours in

Lemma 28, and is the larger one once z_0 exceeds about $4 \cdot 10^6$. In the applications below $4.7 \cdot 10^5 < z_0 < 6.8 \cdot 10^7$, and $c(z_0)$ ranges from about 1.127 to 1.154.

C.5 The shoulder charge

Fix a range $l \le x \le r$ of real bases, and positive numbers p_-, e_-, L_-, L_+ with $p_- \le P$, $e_- \le e^L$ and $L_- \le L \le L_+$ for every x in the range. Since $P \ge \frac{11}{10} x^{1/10} \frac{x}{x+1}$ and $x^{11/10} \le \sqrt{a} \le (x+1)^{11/10}$, one may take $p_- = \frac{11l}{10(l+1)} \rho_-$, $e_- = l\rho_-$, $L_- = \frac{11}{10}[\log l]_-$ and $L_+ = \frac{11}{10}[\log(r+1)]_+$ with $\rho_- = [l^{1/10}]_-$; this is what the computation does. Here and below $[y]_-$ and $[y]_+$ denote rational lower and upper bounds for a real number y, computed as described in Section 4.1. For a strip κ of log-width v_κ and height η_κ put

$$t_\kappa = p_- e_- v_\kappa / U, \qquad z_\kappa = \left\lfloor \sqrt{\lfloor \lambda(\lfloor t_\kappa \rfloor - 1) \rfloor} \right\rfloor,$$

let β_κ be a real number with $\beta_\kappa \le \log(p_- v_\kappa) - \log U + c(z_\kappa)$ and $L_- + \beta_\kappa > 0$, and put $L_\kappa = L_+$ if $\beta_\kappa \ge 0$, $L_\kappa = L_-$ otherwise. (In the computations β_κ is evaluated from below with outward rounding; see Appendix E.)

Proposition 33. *Let G, g and the strips be as in Appendix B, with $v_\kappa < 1$ and $t_\kappa \ge 10^{10}$ for every κ, and with $\lambda(WU+1) \le 4 \cdot 10^{16}$ on the range. Let $\mathcal{S}$ be a set of composite integers m with $1 < |s_m| \le K$ whose prime factors all exceed $2 \cdot 10^8$. Then for $l \le x \le r$,*

$$\frac{\Sigma_+}{P} \le \sum_\kappa \frac{8\eta_\kappa L_\kappa v_\kappa U^3}{L_\kappa + \beta_\kappa} + \frac{4U^2 L_+ \sum_\kappa \eta_\kappa}{e^L P} - \frac{2L}{W} \sum_{m \in \mathcal{S}} g_-(|s_m|).$$

Proof. Apply Lemma 24 with $\mathcal{A}_\kappa^\pm$ the elements of $J_\kappa^\pm$ having no prime factor at most $\sqrt{\lambda N_\kappa^\pm}$, where $N_\kappa^\pm = |J_\kappa^\pm|$. By Lemma 18 the set $J_\kappa^\pm$ consists of the integers of a real interval of length between $W v_\kappa / U \ge t_\kappa$ and $W v_\kappa U$. Hence $N_\kappa^\pm \ge \lfloor t_\kappa \rfloor - 1$, so $z_\kappa \le \sqrt{\lambda N_\kappa^\pm}$; also $z_\kappa \ge \lfloor \sqrt{\lfloor 0.83(10^{10}-1) \rfloor} \rfloor = 91{,}104 \ge 10^4$, as $t_\kappa \ge 10^{10}$; and $N_\kappa^\pm \le W v_\kappa U + 1 < WU + 1$, so $\sqrt{\lambda N_\kappa^\pm} < 2 \cdot 10^8$. In particular $\mathcal{S}$ satisfies the hypothesis of Lemma 24, and every prime of $J_\kappa^\pm$ exceeds $\sqrt{\lambda N_\kappa^\pm}$. By Proposition 32,

$$|\mathcal{A}_\kappa^\pm| \le \frac{2W v_\kappa U}{\log(W v_\kappa / U) + c(z_\kappa)} + 1 \le \frac{2e^L P v_\kappa U}{L + \beta_\kappa} + 1,$$

since $W = e^L P$ and so $\log(W v_\kappa / U) + c(z_\kappa) \ge L + \log(p_- v_\kappa) - \log U + c(z_\kappa) \ge L + \beta_\kappa > 0$. Multiply by $U^2 \cdot 2L/(e^L P)$ and by $2\eta_\kappa$ for the two shoulders.

The function $L \mapsto L/(L+\beta)$ is increasing on $(\max(0,-\beta),\infty)$ if $\beta \geq 0$ and decreasing there if $\beta < 0$, which justifies L_κ. Finally $U^2 \geq 1$ and $2L/(e^L P) = 2L/W$. $\square$

D Semiprimes on the shoulders

In this section we construct the set $\mathcal{S}$ of Proposition 33 and bound $\sum_{m\in\mathcal{S}} g_-(|s_m|)$ from below. The elements of $\mathcal{S}$ are products pq of primes $2\cdot 10^8 < q < p$. For fixed q they correspond to the primes p in a shoulder divided by q. These primes lie at heights between about 4×10^9 and 1.3×10^{20}, where the gaps between consecutive primes have been tabulated exhaustively.

D.1 Prime-gap data

For an integer $b \geq 1$ let $N(b)$ be the number of pairs of consecutive primes $p < p' \leq 4\cdot 10^{18}$ with $p'-p = b$; thus $N(1) = 1$, from the pair $(2,3)$, and $N(b) = 0$ for odd $b \geq 3$. We use two computations, of which only the first has appeared in a refereed journal.

(G1) Oliveira e Silva, Herzog and Pardi [14] computed $N(b)$ for every b. In particular $N(b) = 0$ for $b > 1476$. We also use that $4\cdot 10^{18} - 113$ is prime; the program of Appendix G checks a Pratt certificate for it [16].

(G2) For each row (h_i, b_i) of Table 3, every pair of consecutive primes $p < p'$ with $p < h_i$ satisfies $p'-p \leq b_i$. These bounds are read from the table of maximal prime gaps [19]. Above $4\cdot 10^{18}$ they rest on the completeness of the exhaustive searches for maximal gaps summarised there, up to 2^{64} and from 2^{64} to $1.32\cdot 10^{20}$, which have not appeared in a refereed journal. Only Theorem 6(b) uses (G2).

D.2 Three lemmas on primes in short intervals

For a real interval I we write $|I|$ for its length and $\pi(I)$ for the number of primes in I. For an open interval $I = (u,v)$ and $q > 0$ we write $I/q = (u/q, v/q)$.

i	h_i	b_i	i	h_i	b_i
1	5 733 241 593 241 196 731	1476	7	18 470 057 946 260 698 231	1550
2	6 787 988 999 657 777 797	1488	8	18 571 673 432 051 830 099	1552
3	15 570 628 755 536 096 243	1510	9	20 733 746 510 561 442 863	1572
4	17 678 654 157 568 189 057	1526	10	68 068 810 283 234 182 907	1676
5	18 361 375 334 787 046 697	1530	11	101 412 319 996 363 309 069	1724
6	2^{64}	1550	12	132 000 000 000 000 000 000	1854

Table 3: Bounds for gaps between consecutive primes: every gap whose smaller prime is below h_i has length at most b_i. Except in rows 6 and 12, h_i is the smaller prime of the first gap longer than b_i; row 12 is the end of the exhaustive range.

For a finite family $\mathbf{c} = (c_g)$ of nonnegative reals indexed by positive reals g, put

$$\Phi_{\mathbf{c}}(t) = \sum_g c_g (t-g)_+,$$

$$\Psi_{\mathbf{c}}(t) = \min_{m \geq 1} \big(m - 1 + m\Phi_{\mathbf{c}}(t/m)\big) = \min_{m \geq 1} \Big(m - 1 + \sum_g c_g (t - mg)_+\Big),$$

for $t \geq 0$, and $\Theta_{\mathbf{c}} = \sum_b \Phi_{\mathbf{c}}(b) N(b) = \sum_g c_g \sum_b (b-g)_+ N(b)$. The minimum exists because the bracket is at least $m-1$. The function $\Psi_{\mathbf{c}}$ is nondecreasing, as each bracket is, and $\Psi_{\mathbf{c}}(0) = 0$.

Lemma 34 (Convex gap costs)**.** *Let $I_1, \dots, I_n$ be bounded open intervals contained in $(2, 4 \cdot 10^{18} - 1476)$, any two of which are at distance greater than 1476. Then for every $\mathbf{c}$,*

$$\sum_{i=1}^{n} \pi(I_i) \geq \sum_{i=1}^{n} \Psi_{\mathbf{c}}(|I_i|) - \Theta_{\mathbf{c}}.$$

Proof. Write $I_i = (u_i, v_i)$ and $k_i = \pi(I_i)$. Let p_i^- be the largest prime $\leq u_i$ and p_i^+ the smallest prime $\geq v_i$. The primes in $[p_i^-, p_i^+]$ are p_i^-, the k_i primes of I_i, and p_i^+, so $[p_i^-, p_i^+]$ is the union of $k_i + 1$ *covering gaps* $[p, p']$ between consecutive primes, of total length $p_i^+ - p_i^- \geq v_i - u_i$. Each covering gap $[p, p']$ satisfies $p < v_i$ and $p' > u_i$, so it meets the closure of I_i. Every gap $[p, p']$ between consecutive primes with $p < 4 \cdot 10^{18} - 1476$ has length at most 1476: since $4 \cdot 10^{18} - 113$ is prime, $p' \leq 4 \cdot 10^{18}$, and then (G1) applies. Hence $p_i^+ \leq v_i + 1476 < 4 \cdot 10^{18}$, so every covering gap is counted by N (the gap $[2, 3]$ by $N(1)$). Moreover a gap covering both I_i and $I_{i'}$ with $i \neq i'$

would meet both closures and so have length greater than 1476. Hence the covering gaps of different intervals are different gaps.

Let $d_1, \dots, d_{k_i+1}$ be the lengths of the covering gaps of I_i. The function $\Phi = \Phi_{\mathbf{c}}$ is convex and nondecreasing, so by Jensen's inequality

$$\sum_{j=1}^{k_i+1} \Phi(d_j) \geq (k_i{+}1)\Phi\Big(\frac{d_1 + \cdots + d_{k_i+1}}{k_i+1}\Big) \geq (k_i{+}1)\Phi\Big(\frac{|I_i|}{k_i+1}\Big) \geq \Psi_{\mathbf{c}}(|I_i|){-}k_i,$$

the last step by the definition of $\Psi_{\mathbf{c}}$ with $m = k_i + 1$. Summing over i, and using that the covering gaps are distinct gaps between consecutive primes not exceeding $4 \cdot 10^{18}$, we get $\sum_i \Psi_{\mathbf{c}}(|I_i|) - \sum_i k_i \leq \sum_b \Phi(b)N(b) = \Theta_{\mathbf{c}}$. $\square$

For integer t and integer indices g, $\Psi_{\mathbf{c}}(t)$ is a finite minimum: if $0 \leq t \leq t_{\max}$ and $g \geq g_{\min} > 0$ for all g, the bracket at $m_* = \lfloor t_{\max}/g_{\min} \rfloor + 2$ equals $m_* - 1$, since $m_* g > t$, and every $m > m_*$ gives a bracket at least $m - 1 \geq m_*$. So $\Psi_{\mathbf{c}}(t)$ is the minimum over $1 \leq m \leq m_*$.

Let $M = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$. For an integer $b \geq 100$ define f_b on $\mathbb{Z}$ by

$$f_b(u) = \max\{v \leq u{+}b : \gcd(v, M) = 1\}, \qquad B_b(k) = \max_{\substack{0 \leq s < M \\ \gcd(s,M)=1}} \big(f_b^k(s) - s\big),$$

for $k \geq 0$, where f_b^k is the kth iterate. Consecutive integers prime to M differ by at most 22, so $f_b(u) > u{+}b{-}100$; also f_b is nondecreasing and $f_b(u{+}M) = f_b(u) + M$. Hence B_b is strictly increasing and $B_b(k) > k(b - 100)$.

Lemma 35 (Bounded gaps)**.** *Let (h, b) be a row of Table 3 and $k \geq 1$. Every open interval (u, v) with $u > 2000$, $v \leq h$ and $v - u > B_b(k)$ contains at least k primes.*

Proof. Suppose that (u, v) contains fewer than k primes. Let s_0 be the largest prime $\leq u$ and $s_0 < s_1 < s_2 < \cdots$ the subsequent primes. Then $s_k \geq v$, and we let $i_0 \leq k$ be least with $s_{i_0} \geq v$. For $i < i_0$ we have $s_i < v \leq h$, so $s_{i+1} - s_i \leq b$ by (G2). As $s_{i+1} > 13$ is prime, it is prime to M, so $s_{i+1} \leq f_b(s_i)$. Since f_b is nondecreasing, induction gives $s_{i_0} \leq f_b^{i_0}(s_0)$. Also $s_0 \geq u{-}b > 13$, so s_0 is prime to M, and by periodicity $f_b^{i_0}(s_0) - s_0 \leq B_b(i_0) \leq B_b(k)$. Hence $v \leq s_{i_0} \leq s_0 + B_b(k) \leq u + B_b(k)$, a contradiction. $\square$

Lemma 36. *For $2 \leq y \leq 10^{19}$,*

$$0 < \mathrm{li}(y) - \pi(y) \leq E(y) := \frac{\sqrt{y}}{\log y}\Big(1.95 + \frac{3.9}{\log y} + \frac{19.5}{\log^2 y}\Big).$$

Consequently, for reals $10^8 \le A \le y \le B \le 10^{19}$,

$$\pi(y) - \pi(A) \ge \int_A^y \frac{\mathrm{d}t}{\log t} - E(B).$$

Proof. The first statement is [3, Thm. 2]. For the second, $\pi(y) - \pi(A) > \mathrm{li}(y) - E(y) - \mathrm{li}(A)$, and E is increasing on $[e^6, \infty)$ because $\sqrt{y}/\log^j y$ is increasing for $\log y > 2j$. □

D.3 The windows

Fix a range $l \le x \le r$ of real bases. Let $\rho_- = [l^{1/10}]_-$ and $\rho_+ = [(r+1)^{1/10}]_+$, and put

$$A_- = 0.999999\,(l\rho_-)^2, \qquad A_+ = 1.000001\,((r+1)\rho_+)^2,$$
$$W_- = \Big\lfloor 0.999999\,\frac{11l}{10(l+1)}\,l\rho_-^2 \Big\rfloor, \qquad W_+ = \Big\lfloor \frac{11(r+1)}{10r}\,(r+1)\rho_+^2 \Big\rfloor + 1,$$

$Q = 1.5 \cdot 10^{13}$, and let q_0 be the least multiple of 10^8 exceeding $\lfloor A_+/(4 \cdot 10^{18} - 1500) \rfloor$. Write $v_j = 7j/(25 \cdot 3072)$ for $1 \le j \le 3072$, so $v_{3072} = 0.28$, and for x in the range let

$$J_j^+ = (ae^D, ae^{D(1+v_j)}), \qquad J_j^- = (ae^{-D(1+v_j)}, ae^{-D}).$$

An integer m satisfies $1 < \pm s_m \le 1 + v_j$ if and only if $m \in J_j^\pm$ or m is the right, respectively left, endpoint of $J_j^\pm$.

Lemma 37. *Let* $l \le x \le r$ *with* $l \ge 10^{13}$, $r \le 10^{17}$. *Then* $W_- \le W/U$ *and* $W \le W_+$; $|J_j^\pm| \ge W_- v_j$; *every* m *with* $1 < |s_m| \le 1.28$ *lies in* (A_-, A_+); *and* $a(e^{1.28D} - e^{-1.28D}) \le 2.56001 \cdot 1.000001\, W_+$.

Proof. Since $\log(1 + 1/x) \ge 1/(x+1)$ we have $D \ge 1.1/(x+1)$ and $W = aD \ge 1.1x^{2.2}/(x+1)$, which is increasing in x, so $W \ge 1.1\,l^{2.2}/(l+1) \ge W_-/0.999999 > UW_-$. Since $\log(1 + 1/x) \le 1/x$, $W \le 1.1x^{0.1}(x+1)^{1.1} \le 1.1(r+1)^{2.2}/r < W_+$. By Lemma 18, $|J_j^\pm| \ge Wv_j/U \ge W_- v_j$. Every m with $1 < |s_m| \le 1.28$ lies in $[ae^{-1.28D}, ae^{1.28D}] \subset [a/U, aU]$, and $x^{2.2} \le a \le (x+1)^{2.2}$. Finally $2\sinh(1.28D) \le 2.56001D$ as $D < 10^{-12}$. □

D.4 The deduction

Let g_- be a function on $(1, K]$ of the form

$$g_- = \sum_{j=1}^{3072} \delta_j \mathbf{1}_{(1,1+v_j]}, \qquad \delta_j \ge 0, \tag{14}$$

with $0 \le g_- \le G_+$; it is nonincreasing and vanishes on $(1.28, K]$. Its construction is in Appendix E. For each j let $\mathbf{c}^{(j)}$ be a family as in Lemma 34 (in the computation, one of 48 fixed families), and put

$$H_j = \lfloor W_- v_j \rfloor, \qquad H_j' = \lceil W_- v_j \rceil - 1,$$

so that $H_j' < W_- v_j$. Define record bands as follows. Put $u_0 = q_0$. For $i = 1, \dots, 12$ let $q_i^- = \lceil A_+/(h_i - 2000) \rceil$; if $q_i^- < u_{i-1}$ the ith band is $(q_i^-, q_i^+]$ with $q_i^+ = u_{i-1}$, and we set $u_i = q_i^-$; otherwise the band is empty and $u_i = u_{i-1}$. Let $q_* = u_{12}$. Finally let $\pi_0^\flat$ and $\pi_i^\flat$ be nondecreasing functions on $[0, \infty)$ with

$$\begin{aligned} 0 \le \pi_0^\flat(y) &\le \big[\pi(\min(y, Q)) - \pi(q_0)\big]_+, \\ 0 \le \pi_i^\flat(y) &\le \big[\pi(\min(y, q_i^+)) - \pi(q_i^-)\big]_+. \end{aligned} \tag{15}$$

(The right sides vanish for $y < q_0$, respectively $y < q_i^-$, so $\pi_0^\flat$ and $\pi_i^\flat$ vanish on $[0, 1)$; this is how the arguments $\lfloor H_j/t \rfloor = 0$ and $\lfloor H_j'/B_{b_i}(k) \rfloor = 0$ below are treated. One may take $\pi_i^\flat = 0$, and $\pi_0^\flat = 0$ as well; then $\mathfrak{d}_{\mathrm{rec}}$, respectively $\mathfrak{d}_{\mathrm{hist}}$, below vanishes.)

Proposition 38. *Suppose that* $10^{13} \le l < r \le 10^{17}$ *and*

$$\begin{gathered} q_0 < Q, \qquad \frac{2W_-}{Q} > 1476, \qquad \frac{A_-}{Q} > Q + 1476, \qquad \frac{A_+}{q_0} + 1476 < 4 \cdot 10^{18}, \\ 2A_- - 2.56001 \cdot 1.000001\, W_+ Q > 1476\, Q(Q+2), \\ q_i^- > 2 \cdot 10^8 \text{ for every nonempty band.} \end{gathered} \tag{16}$$

Let $\mathcal{S}$ be the set of products pq of primes $q_* < q \le Q$ and $p > q$ with $1 < |s_{pq}| \le 1.28$. Then $\mathcal{S}$ is a set of composite integers m with $1 < |s_m| \le K$ whose prime factors exceed $2 \cdot 10^8$, and for every $x \in [l, r]$

$$\frac{2L}{W} \sum_{m \in \mathcal{S}} g_-(|s_m|) \ge \mathfrak{d}_{\mathrm{hist}} + \mathfrak{d}_{\mathrm{rec}},$$

where, with $\nu = 0.999999 \cdot 2.2\, [\log l]_-/W_+$ and $\Delta\Psi(t) = \Psi(t) - \Psi(t-1)$,

$$\begin{aligned} \mathfrak{d}_{\mathrm{hist}} &= \nu \sum_{j=1}^{3072} \delta_j \max\Big(0,\ 2 \sum_{t \ge 1} \Delta\Psi_{\mathbf{c}^{(j)}}(t)\, \pi_0^\flat\big(\lfloor H_j/t \rfloor\big) - \Theta_{\mathbf{c}^{(j)}}\Big), \\ \mathfrak{d}_{\mathrm{rec}} &= \nu \sum_{j=1}^{3072} \delta_j \cdot 2 \sum_i \sum_{k \ge 1} \pi_i^\flat\big(\lfloor H_j'/B_{b_i}(k) \rfloor\big), \end{aligned}$$

the sum over i running over the nonempty bands. All the sums are finite.

Proof. Every q in the statement exceeds $q_* \ge \min_i q_i^- > 2\cdot 10^8$ (or $q_0 > 2\cdot 10^8$ if all bands are empty). For $pq \in \mathcal{S}$, Lemma 37 gives $pq > A_-$, so $p > A_-/Q > Q \ge q$; thus q is the least prime factor of pq, the map $(p,q)\mapsto pq$ is injective, and $\mathcal{S}$ has the stated properties.

By (14), $\sum_{m\in\mathcal{S}} g_-(|s_m|) = \sum_j \delta_j \,\#\{m\in\mathcal{S} : 1 < |s_m| \le 1+v_j\}$. For fixed j and a prime q in $(q_*, Q]$, every prime $p \in J_j^\pm/q$ gives an element pq of $\mathcal{S}$ with $1 < |s_{pq}| \le 1+v_j$, because $pq \in J_j^\pm \subset (A_-, A_+)$ by Lemma 37, so $p > A_-/Q > q$. Hence

$$\#\{m\in\mathcal{S} : 1 < |s_m| \le 1+v_j\} \\ \ge \sum_{\pm}\sum_{q_0<q\le Q} \pi(J_j^\pm/q) + \sum_{\pm}\sum_i \sum_{q_i^-<q\le q_i^+} \pi(J_j^\pm/q). \quad (17)$$

Since $2L = \log a \ge 2.2\log l$ and $W \le W_+$, we have $2L/W \ge \nu$. It remains to bound the two parts of (17).

Histogram band. Consider the $2(\pi(Q)-\pi(q_0))$ open intervals $J_j^\pm/q$ with $q_0 < q \le Q$ prime. By Lemma 37 each lies in $[A_-/Q, A_+/q_0] \subset (Q+1476, 4\cdot 10^{18} - 1476)$. For one q, the two intervals are at distance at least $a(e^D - e^{-D})/q \ge 2W/q \ge 2W_-/Q > 1476$. For primes $q < q'$ in the band, every point of $J_j^\pm/q$ is at least a_1/q and every point of $J_j^\pm/q'$ is at most a_2/q', where $a_1 = ae^{-1.28D}$, $a_2 = ae^{1.28D}$. Since $q' \ge q+2$,

$$\frac{a_1}{q} - \frac{a_2}{q'} = \frac{a_1(q'-q) - (a_2-a_1)q}{qq'} \\ \ge \frac{2A_- - 2.56001\cdot 1.000001\, W_+ Q}{Q^2} > 1476,$$

by Lemma 37 and (16). So Lemma 34 applies, and since Ψ is nondecreasing and $|J_j^\pm/q| \ge W_- v_j/q \ge \lfloor H_j/q\rfloor$,

$$\sum_{\pm}\sum_{q_0<q\le Q} \pi(J_j^\pm/q) \ge 2\sum_{q_0<q\le Q} \Psi_{\mathbf{c}^{(j)}}\big(\lfloor H_j/q\rfloor\big) - \Theta_{\mathbf{c}^{(j)}}.$$

Writing $\Psi(n) = \sum_{t=1}^n \Delta\Psi(t)$ with $\Delta\Psi \ge 0$ and exchanging the sums,

$$\sum_{q_0<q\le Q} \Psi\big(\lfloor H_j/q\rfloor\big) = \sum_{t\ge 1}\Delta\Psi(t)\,\#\{q_0 < q \le Q :\ q \le \lfloor H_j/t\rfloor\} \\ \ge \sum_{t\ge 1}\Delta\Psi(t)\,\pi_0^\flat\big(\lfloor H_j/t\rfloor\big).$$

The left side of the previous display is also nonnegative, which gives the term of $\mathfrak{d}_{\mathrm{hist}}$. Only $t \le H_j/q_0$ contribute.

Record bands. Let q be a prime in the ith band. The interval $J_j^{\pm}/q$ has length at least $W_- v_j/q > H_j'/q$, its left end exceeds $A_-/Q > 2000$, and its right end is at most $A_+/q < A_+/q_i^- \le h_i - 2000$. By Lemma 35, it contains at least k primes whenever $qB_{b_i}(k) \le H_j'$. Since B_{b_i} is increasing,

$$\begin{aligned}\sum_{q_i^- < q \le q_i^+} \pi(J_j^{\pm}/q) &\ge \sum_{k \ge 1} \#\bigl\{q_i^- < q \le q_i^+ :\ q \le \lfloor H_j'/B_{b_i}(k) \rfloor\bigr\} \\ &\ge \sum_{k \ge 1} \pi_i^{\flat}\bigl(\lfloor H_j'/B_{b_i}(k) \rfloor\bigr).\end{aligned}$$

The terms vanish once $B_{b_i}(k) > H_j'/q_i^-$, which happens for $k > H_j'/(q_i^-(b_i - 100))$. Summing over both signs and over the bands gives the term of $\mathfrak{d}_{\mathrm{rec}}$. □

Remark (Evaluation). In the computation $\rho_{\pm}$ are the enclosures of Section 4.1, which are multiples of 2^{-200}. The functions $\pi^{\flat}$ come from Lemma 36: $\pi_0^{\flat}(y) = \max\bigl(0, \lfloor S_0(\min(y, Q)) \rfloor - \lceil E(Q) \rceil\bigr)$, where $S_0(y)$ is a lower Riemann sum for $\int_{q_0}^{y} \mathrm{d}t/\log t$ on the mesh $q_0 + 10^9\mathbb{Z}$, with the value of $1/\log$ at the right end of each mesh cell rounded down to a multiple of 10^{-6}; $\pi_i^{\flat}$ is defined in the same way on $[q_i^-, q_i^+]$ with mesh 10^7, values rounded down to multiples of 10^{-10}, and $\lceil E(q_i^+) \rceil$. Here $\lceil E(\cdot) \rceil$ is evaluated with $\sqrt{y} \le \lfloor \sqrt{y} \rfloor + 1$ and a lower bound for $\log y$. The families $\mathbf{c}^{(j)}$ are taken from a list of 48 families $\mathbf{c}_0, \dots, \mathbf{c}_{47}$, with $\mathbf{c}^{(j)} = \mathbf{c}_{\min(47, \lfloor 48j/3072 \rfloor)}$; every index g is an integer in $[270, 900]$, every c_g is a nonnegative multiple of 10^{-6}, and each family has at most six terms. They were found by linear programming, but Lemma 34 holds for any choice. With integer t and g, all values of $\Psi_{\mathbf{c}}$ are multiples of 10^{-6} and are computed exactly from the remark after Lemma 34. The numbers $\Theta_{\mathbf{c}}$ are computed exactly from the table (G1), and $B_b(k)$ exactly by iterating f_b from each of the $\varphi(M) = 5760$ residues. The conditions (16) are checked for each range in Table 5; there q_0 lies between $1.345 \cdot 10^{11}$ and $1.988 \cdot 10^{11}$.

E The twenty-mode test

Let G_{20} be the cosine test of support $K = \frac{5}{2}$ with the twenty weights $w_j = \omega_j/10^9$ of Table 4, and $F_{20} = \widehat{G_{20}}$. Then $\sum |w_j| = 0.761870883 < 1$ and $\|G_{20}\|_1 \le 2K \sum |w_j| < 4$.

j	ω_j	j	ω_j	j	ω_j	j	ω_j
0	380 707 290	5	−1 646 098	10	210 054	15	1 271
1	249 434 692	6	6 624 743	11	19 455	16	−226
2	84 682 233	7	1 562 594	12	−8 325	17	85
3	−12 031 630	8	−1 500 312	13	1 797	18	−52
4	−23 007 900	9	−429 738	14	−2 362	19	26

Table 4: The twenty-mode test: $G_{20}(s) = \mathbf{1}_{[-5/2,5/2]}(s)\sum_j 10^{-9}\omega_j \cos(\pi(2j+1)s/5)$.

Lemma 39. *With* $N_{20} = 0.761942143419865$ *and* $M_{20} = 1.377680396761364$,

$$F_{20}(0) > 1.000000000974554, \qquad \|F_{20}\|_1 < N_{20}, \qquad M_{F_{20}} + 10^{-8} \le -M_{20},$$
$$V_0 < 514, \qquad V_1 < 1000,$$

and $F_{20}(y) \ge 1$ *for* $|y| \le 7\cdot 10^{-9}$.

Proof. Write $F = F_{20}$, $\kappa_r = \sum_j k_j^r |w_j|$, and recall $F(0) = \frac{4K}{\pi}\sum_j (-1)^j w_j/k_j$ from (2); the first bound follows from the enclosure of π. Since $F^{(n)}(y) = \int (2\pi i s)^n G(s) e^{2\pi i y s}\,\mathrm{d}s$, we have $|F^{(n)}| \le (2\pi)^n \frac{2K^{n+1}}{n+1}\sum |w_j|$. For $n = 1, 2, 4$ this gives $|F'| < 32$, $|F''| < 328$ and $|F^{(4)}| \le M_4 := (2\pi)^4 \frac{2K^5}{5}\sum |w_j|$. In particular $F(y) \ge F(0) - 164y^2 > 1$ for $|y| \le 7\cdot 10^{-9}$.

Tails. For $y \ge 16$ put $X = 4Ky \ge 160$ and $\lambda_0 = 1 - (39/160)^2$, so that $|k_j^2 - X^2| \ge \lambda_0 X^2$ for all j. Expanding $1/(k^2 - X^2) = -\sum_{i=0}^{2} k^{2i} X^{-2i-2} - k^6 X^{-6}(X^2 - k^2)^{-1}$ in (2) gives

$$|F(y)| \le \sum_{r\in\{1,3,5\}} \frac{c_r}{y^{r+1}} + \frac{c_7}{y^8},$$
$$c_r = \frac{|\sum_j (-1)^j k_j^r w_j|}{\pi(4K)^r} \quad (r \le 5), \qquad c_7 = \frac{\kappa_7}{\pi(4K)^7\lambda_0}.$$

Differentiating (2) directly and using the same lower bound for $|k_j^2 - X^2|$,

$$|F'(y)| \le \frac{4K}{\pi}\cdot\frac{\kappa_1}{(4K)^2}\Big(\frac{2\pi K}{\lambda_0 y^2} + \frac{2}{\lambda_0^2 y^3}\Big) \le \frac{C_1}{y^2},$$
$$C_1 = \frac{4K}{\pi}\cdot\frac{\kappa_1}{(4K)^2}\Big(\frac{2\pi K}{\lambda_0} + \frac{2}{16\lambda_0^2}\Big) < 2.$$

Hence $V_0 \le 16\cdot 32 + C_1/16 < 514$ and $V_1 \le 32(16\log 16 - 15) + C_1(\log 16 + 1)/16 < 1000$.

The integrals on $[0,16]$. Let $h = 1/256$. The values of F at the points $i/768$, $0 \le i \le 12288$, are enclosed using (2), and the midpoints of the enclosures, rounded to multiples of 10^{-18}, are within 10^{-14} of the true values. On each panel $[jh,(j+1)h]$ let p_j be the cubic polynomial interpolating these rounded values at the four points $jh + ih/3$, and let $b_0,\dots,b_3$ be its Bernstein coefficients on the panel, so that $|p_j| \le \sum_i |b_i| B_i$ where B_i is the ith Bernstein basis polynomial and $\int_{\text{panel}} B_i = h/4$. The interpolation error is at most $M_4(h/3)^4/24$, because $|\tau(\tau-1)(\tau-2)(\tau-3)| \le 1$ on $[0,3]$. The rounding of the four values moves each b_i by at most $\frac{34}{6}\cdot 10^{-14}$. Hence, with $\varepsilon = M_4 h^4/1944 + 10^{-13}$,

$$\int_{jh}^{(j+1)h} |F| \le \frac{h}{4}\sum_i |b_i| + \varepsilon h.$$

For the logarithmic moment, on panels with $j \ge 256$ (where $\log y \ge 0$) we use $|F| \le \sum_i |b_i| B_i + \varepsilon$; on panels with $j < 256$ (where $\log y < 0$) we use $|F| \ge \sum_i |b_i| B_i - \varepsilon$ when all b_i have the same sign, and $|F|\log y \le 0$ otherwise. The integrals $\int B_i(y)\log y \,\mathrm{d}y$ over each panel are evaluated in closed form from enclosures of $\log(jh)$ with error below 10^{-40}; the closed forms amplify this error by less than $3(j+1)^4$, which is absorbed by adding 10^{-18}. The interpolation error contributes at most $\varepsilon\int_0^{16}|\log y|\,\mathrm{d}y \le \varepsilon(2 + 16\log 16)$. Adding twice the integrals over $[0,16]$ (as F is even) and twice the tail integrals $\int_{16}^{\infty} c/y^{r+1} = c/(r16^r)$ and $\int_{16}^{\infty} c\log y/y^{r+1} = c(\log 16 + 1/r)/(r16^r)$ gives $\|F\|_1 < 0.76194214342$ and $M_F < -1.37768040676$. The last two digits displayed are enough: the program checks $\|F\|_1 < N_{20}$ and $M_F + 10^{-8} \le -M_{20}$ exactly. □

Lemma 40. *There is a majorant* $g \ge (G_{20})_+$ *on* $(1,\frac{5}{2}]$ *of the form considered in Appendix B with* $M = 65536$ *cells per unit, whose strips satisfy* $v_\kappa < 1$ *and* $\sum_\kappa \eta_\kappa < 0.14$. *There is a minorant* g_- *of the form* (14) *with* $0 \le g_- \le (G_{20})_+$.

Proof. Put $\kappa = \sum |w_j|(\pi k_j/(2K))^2$, an upper bound for $|G_{20}''|$. The values $G_{20}(1+i/65536)$, $0 \le i \le 98304$, are computed in fixed-point arithmetic with denominator 10^{22}, generating $\cos(\pi k(1+i/65536)/(2K))$ and the corresponding sines for each mode by the rotation recurrence. The starting values and the rotation coefficients are rounded from enclosures with error below $3\cdot 10^{-18}$, so a rotation step multiplies the existing error by at most $1 + 5\cdot 10^{-18}$ and adds at most $6\cdot 10^{-18}$; after $n \le 98304$ steps the error is below $12\cdot 10^{-18}(n+1) < 2\cdot 10^{-12}$, also after weighting by the w_j. On the cell between consecutive nodes G_{20} is at most the larger computed endpoint

value plus $\kappa/(8\cdot 65536^2)+2\cdot 10^{-12}$, the last term for the node error. Rounding these upper values up to multiples of 10^{-28}, and replacing negative values by 0, gives the heights h_i of g. The strips are produced by the stack algorithm of Appendix B; the program checks (8), that every strip has fewer than 65536 cells, and that $\sum_\kappa \eta_\kappa = 0.12675\ldots < 0.14$.

For g_- use the 3072 cells $(1+v_{j-1}, 1+v_j]$ of width $v_1 = 7/(25\cdot 3072)$. On each, G_{20} is at least the smaller endpoint value minus $\kappa v_1^2/8$; subtract a further $2\cdot 10^{-12}$ for the node error, round down to a multiple of 10^{-12}, and replace negative values by 0. Let γ_j be the minimum of these values over the first j cells. Then $g_- = \gamma_j$ on the jth cell is nonincreasing, $0 \le g_- \le (G_{20})_+$, and $g_- = \sum_j \delta_j \mathbf{1}_{(1,1+v_j]}$ with $\delta_j = \gamma_j - \gamma_{j+1} \ge 0$ ($\gamma_{3073} = 0$). □

F Proof of Theorem 6

We show that no $x \in [x_0, 3.9\cdot 10^{13}]$ is a hole if (G2) holds (Proposition 42), and that no $x \in [32{,}746{,}701{,}927{,}355,\ 3.9\cdot 10^{13}]$ is a hole without (G2) (Proposition 43); with Theorem 4 this proves Theorem 6. The notation is that of Section 4; in particular (7) holds.

In the rest of this section $G = G_{20}$, $K = \frac{5}{2}$, N_{20} and M_{20} are as in Lemma 39, and for $k \ge 0$ we put

$$J_k(x) = 1.1\,k\,x^{1/10}\frac{x}{x+1} - N_{20}\log\frac{x+1}{1.1} + M_{20}.$$

Let $\gamma_1 < \gamma_2 < \cdots < \gamma_{1\,400\,000}$ be the first 1.4×10^6 positive ordinates in the list of Platt [15, 10], all below 816,393, and put

$$S(u) = \sum_{j=1}^{1\,400\,000} \big(1-\cos(\gamma_j u)\big), \qquad u(x) = \log a = 1.1\log(x(x+1)).$$

Lemma 41. *Let $10^{13} \le l < r \le 10^{14}$, let $x \in [l,r]$, and suppose that, with $\rho_\pm$, $A_\pm$, $W_\pm$, q_0 and Q as in Appendix D,*

(i) $x \ge 3.2\cdot 10^{13}$ and (a,D) satisfies (4) with $K = \frac{5}{2}$;

(ii) at x, the sum $E_{\mathrm{pow}} + 2D[(3 + 0.15\log\frac{2\pi}{D})V_0 + 0.15V_1 + 4] + 4U^2L_+\sum_\kappa \eta_\kappa/e^L + 10^{-12}$ is less than 10^{-8};

(iii) the hypotheses of Proposition 33 hold for the strips of Lemma 40 with $p_-, e_-, L_\pm$ as in Appendix C; and either those of Proposition 38 hold, or $\mathfrak{d}_{\mathrm{hist}} = \mathfrak{d}_{\mathrm{rec}} = 0$;

(iv) $0 \le k \le F_{20}(0) - \mathfrak{c} + \mathfrak{d}_{\mathrm{hist}} + \mathfrak{d}_{\mathrm{rec}}$, *where* $\mathfrak{c} = \sum_\kappa 8\eta_\kappa L_\kappa v_\kappa U^3/(L_\kappa + \beta_\kappa)$.

If x is a hole, then $J_k(x) < 0$. If, moreover, $\gamma_1, \dots, \gamma_{1\,400\,000}$ are ordinates of zeros of ζ, then $J_k(x) + 2.2S(u(x))/(x+1) < 0$.

Proof. By (i) we may apply Proposition 17, with $\mathcal{B} = \emptyset$ for the first statement and with $\mathcal{B} = \{\gamma_1, \dots, \gamma_{1\,400\,000}\}$ for the second. With $\mathcal{B} = \emptyset$ the zero term in (5) vanishes; put $S^* = 0$. In the second case $\Delta\gamma_j \le 1.1 \cdot 816393/(2\pi x) < 7 \cdot 10^{-9}$, so $F(\Delta\gamma_j) \ge 1$ by Lemma 39, and the zero term is at most $-2DS(u(x)) \le -2.2S^*/(x+1)$ with $S^* = S(u(x))$. If $\mathfrak{d}_{\mathrm{hist}} = \mathfrak{d}_{\mathrm{rec}} = 0$ we apply Proposition 33 with $\mathcal{S} = \emptyset$; otherwise we apply it with the set $\mathcal{S}$ of Proposition 38, and then Proposition 38. In either case

$$\Sigma_+ \le P\big(\mathfrak{c} - \mathfrak{d}_{\mathrm{hist}} - \mathfrak{d}_{\mathrm{rec}}\big) + 4U^2 L_+ \sum_\kappa \eta_\kappa / e^L.$$

With (ii), (5) becomes $Pk \le P(F(0) - \mathfrak{c} + \mathfrak{d}_{\mathrm{hist}} + \mathfrak{d}_{\mathrm{rec}}) < \|F\|_1 \log\frac{1}{D} + M_F + 10^{-8} - 2.2S^*/(x+1)$. By Lemma 39 and (7) the left side is at least $1.1kx^{1/10}x/(x+1)$ and the right side is less than $N_{20}\log\frac{x+1}{1.1} - M_{20} - 2.2S^*/(x+1)$. □

Proposition 42. *Assume (G2). Then no $x \in [x_0, 3.9 \cdot 10^{13}]$ is a hole.*

Proof. Put $l_* = 32{,}654{,}400{,}000{,}000$ and $r_* = 3.9 \cdot 10^{13}$. The program checks (i) and (ii) of Lemma 41 once for all $x \in [l_*, r_*]$. For (i), $\log a \ge 2.2 \log l_* > 68$, and the program checks $KD \le 2.75/l_* < (U-1)/U$ and $KP \le 2.75(r_*+1)[(r_*+1)^{1/10}]_+/r_* < 66$, using (7) and that $(x+1)^{11/10}/x$ is increasing. For (ii) it uses $V_0 < 514$, $V_1 < 1000$, $\ell < 71$, $\ell/\log 2 < 103$ and $1 + KPU < 67$, which give $E_{\mathrm{pow}} \le (103 \cdot 67 + 2) \cdot 71U/(l_*[l_*^{1/10}]_-)$; it bounds the zero term using $D \le 1.1/l_*$ and $\log(2\pi/D) \le \log(7(r_*+1)/1.1)$, and checks $4U^2L_+ \cdot 0.14/(l\rho_-) < 10^{-9}$ for each row, which bounds the third term as $\sum_\kappa \eta_\kappa < 0.14$ and $e^L \ge l\rho_-$; the four terms total less than 10^{-8}. For each row it checks (iii) and (iv), and evaluates $\mathfrak{c}$, $\mathfrak{d}_{\mathrm{hist}}$, $\mathfrak{d}_{\mathrm{rec}}$ with outward rounding: β_κ is computed as $[\log(p_- v_\kappa)]_- + c_-(z_\kappa) - [\log U]_+$, where $c_-(z)$ is the expression $c(z)$ of Proposition 32 with C_0 replaced by the lower bound computed in Lemma 31 and $\sqrt{z}$ replaced by $\lfloor\sqrt{z}\rfloor$, and every term of $\mathfrak{c}$ is rounded up to a multiple of 10^{-21}. It also checks that

$$0.11\, k\, \rho_- \frac{l}{l+1} - N_{20} > 1.4. \tag{18}$$

For $x \ge l$ we have $\frac{\mathrm{d}}{\mathrm{d}\log x} J_k(x) \ge 0.11kx^{1/10}\frac{x}{x+1} - N_{20}x/(x+1) > 1.4$ by (18). The program finds the least integer $n_* \ge l$ with $J_k(n_*) > 0$ (by bisection,

	l	r	n_*
1	32 654 400 000 000	32 654 591 400 000	32 654 580 892 712
2	32 654 000 000 000	32 654 728 300 000	32 654 591 385 176
3	32 650 000 000 000	32 657 973 000 000	32 654 728 287 294
4	32 657 900 000 000	32 662 000 000 000	l
5	32 661 000 000 000	32 700 000 000 000	l
6	32 690 000 000 000	32 800 000 000 000	l
7	32 790 000 000 000	33 116 944 888 557	l
8	33 100 000 000 000	33 500 000 000 000	l
9	33 500 000 000 000	35 000 000 000 000	l
10	35 000 000 000 000	39 000 000 000 000	l

	$\mathfrak{c}$	$\mathfrak{d}_{\mathrm{rec}}$	$\mathfrak{d}_{\mathrm{hist}}$	k
1	0.09958069	0.00016341	0.00040665	0.900989374141681
2	0.09958069	0.00016340	0.00040664	0.900989355101264
3	0.09958075	0.00016335	0.00040650	0.900989106668924
4	0.09958071	0.00016339	0.00040650	0.900989189387311
5	0.09958092	0.00016314	0.00040565	0.900987876087870
6	0.09958130	0.00016266	0.00040375	0.900985113136582
7	0.09958243	0.00016121	0.00039798	0.900976756964004
8	0.09958221	0.00016067	0.00039518	0.900973639434988
9	0.09958788	0.00015388	0.00036858	0.900934574830139
10	0.09959798	0.00014072	0.00031735	0.900860090222618

Table 5: The ten ranges of Proposition 42. The values of $\mathfrak{c}$, $\mathfrak{d}_{\mathrm{rec}}$, $\mathfrak{d}_{\mathrm{hist}}$ are rounded for display; k is the exact value used, obtained by rounding $F_{20}(0)_- - \mathfrak{c} + \mathfrak{d}_{\mathrm{rec}} + \mathfrak{d}_{\mathrm{hist}}$ down to a multiple of 10^{-15}, where $F_{20}(0)_-$ is the computed lower bound for $F_{20}(0)$. The last column is the least integer $n_* \geq l$ with $J_k(n_*) > 0$; for rows 4–10 it is l itself.

checking $J_k(n_* - 1) < 0$ when $n_* > l$), so $J_k > 0$ on $[n_*, \infty)$. Every n_* is at least l_*, so by Lemma 41 no $x \in [n_*, r]$ is a hole. In Table 5 each n_* is at most the r of the previous row, so the rows cover $[n_*^{(1)}, r_*]$ with $n_*^{(1)} = 32{,}654{,}580{,}892{,}712$.

It remains to treat $[x_0, n_*^{(1)}]$, with the first row of Table 5, using the zeros. Put $r_0 = n_*^{(1)} - 2{,}100{,}000 = 32{,}654{,}578{,}792{,}712$ and $u_0 = u(r_0)$. The program computes

$$1{,}400{,}961.5773 < S(u_0) < 1{,}400{,}961.5922, \qquad |S'(u_0)| < 205{,}964{,}368,$$

as described below. Since $|u'(x)| \le 2.2/x$, we have $|u(x_0) - u_0| \le \delta := 2.2(r_0 - x_0)/x_0$, and $|S''| \le \sum_j \gamma_j^2 \le 1.4 \cdot 10^6 \cdot 816393^2$. By Taylor's theorem

$$S(u(x_0)) \ge S(u_0) - |S'(u_0)|\delta - \tfrac{1}{2} \cdot 1.4 \cdot 10^6 \cdot 816393^2\, \delta^2 > 1{,}400{,}961.44.$$

Let $\Phi(x) = J_k(x) + 2.2S(u(x))/(x+1)$. The program checks $\Phi(x_0) > 2.99 \cdot 10^{-14}$ using a lower bound for $J_k(x_0)$. For $x \ge x_0$, as $0 \le S \le 2.8 \cdot 10^6$ and $|S'| \le 1.4 \cdot 10^6 \cdot 816393$,

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\log x}\frac{2.2S(u(x))}{x+1}\Big| \le \frac{2.2}{x} \cdot 1.4 \cdot 10^6\,(2 + 2.2 \cdot 816393) < 0.34,$$

so Φ is increasing on $[x_0, \infty)$ by (18), and $\Phi > 0$ there. By Lemma 41 no $x \in [x_0, r]$ is a hole, where $r = 32{,}654{,}591{,}400{,}000$ is the right end of the first row.

Evaluation of $S(u_0)$ and $S'(u_0)$. The archived list gives each γ_j truncated to 11 decimal places. Platt's ordinates are certified to within 2^{-102}, and the LMFDB display retains at least 25 decimals, so each archived value $\tilde\gamma_j$ satisfies $|\gamma_j - \tilde\gamma_j| < 1.00001 \cdot 10^{-11}$; consecutive archived values differ by more than $2 \cdot 10^{-11}$, so they belong to distinct zeros, and all are below 816,393. The phase $\tilde\gamma_j u_0/(2\pi)$ is reduced modulo 1 in fixed point with denominator 2^{96}, using an enclosure of $u_0/(2\pi)$; cos is evaluated by linear interpolation in a table of 32768 rigorous upper bounds for $\cos(2\pi i/32768)$, rounded to a fixed denominator. The total error per term is at most

$$10^{-16} + [u_0]_+ \cdot 1.00001 \cdot 10^{-11} + 2\pi \cdot 2 \cdot 816393 \cdot 10^{11}/2^{96} + (2\pi)^2/(8 \cdot 32768^2),$$

from the table, the ordinates, the phase reduction and the interpolation; $S'(u_0) = \sum_j \gamma_j \sin(\gamma_j u_0)$ is bounded in the same way. □

Proposition 43. *No $x \in [32{,}746{,}701{,}927{,}355,\ 3.9 \cdot 10^{13}]$ is a hole. Without (G1), the same holds for $x \in [32{,}971{,}281{,}678{,}698,\ 3.9 \cdot 10^{13}]$. Neither statement uses (G2) or the zeros.*

Proof. We use the rows of Table 5 and the first statement of Lemma 41; its conditions (i) and (ii) were checked in the proof of Proposition 42. For the first statement take $\pi_i^\flat = 0$ for every band, so that $\mathfrak{d}_{\mathrm{rec}} = 0$, and keep $\pi_0^\flat$ and the families $\mathbf{c}^{(j)}$ of the remark on evaluation in Appendix D.4; then Proposition 38 uses only (G1), through Lemma 34. For the second take $\mathfrak{d}_{\mathrm{hist}} = \mathfrak{d}_{\mathrm{rec}} = 0$. For each row the program recomputes k as in Table 5, now by rounding $F_{20}(0)_- - \mathfrak{c} + \mathfrak{d}_{\mathrm{hist}}$, respectively $F_{20}(0)_- - \mathfrak{c}$, down to a multiple of 10^{-15}. It checks (18), and finds the least integer $n_* \ge l$ with

$J_k(n_*) > 0$ whenever $J_k(r) > 0$. In the first case rows 1–5 give no n_*, row 6 gives $k = 0.900822450956687$ and $n_* = 32{,}746{,}701{,}927{,}355$, and rows 7–10 have $n_* = l$. In the second case rows 1–6 give no n_*, row 7 gives $k = 0.900417566255251$ and $n_* = 32{,}971{,}281{,}678{,}698$, and rows 8–10 have $n_* = l$. In both cases each of these n_* is at most the r of the preceding row used, so, as in the proof of Proposition 42, the rows cover the stated ranges. □

Proof of Theorem 6. By Proposition 43, and by Proposition 42 under (G2), no x in the stated ranges up to $3.9 \cdot 10^{13}$ is a hole, and by Theorem 4 no $x \geq 35 \cdot 10^{12}$ is a hole. □

Remark. Without the zeros, Proposition 42 would start at $32{,}654{,}580{,}892{,}712$, so the zeros lower the cutoff of Theorem 6(b) by $2{,}105{,}322$. Without the deductions of Appendix D, the coefficient k on the first row would be smaller by $5.7 \cdot 10^{-4}$. With only the rows of Table 3 with $h_i \leq 2^{64}$ (those above $4 \cdot 10^{18}$ still rest on the search up to 2^{64}), together with (G1) and without the zeros, the same slabs give $32{,}706{,}763{,}162{,}237$.

G Programs and data

All programs and data are available at `https://github.com/charlieoneill11/maths-releases`. The analytic statements are checked by Python programs, run with assertions enabled.

Theorem 4. One program, which uses only the Python standard library and takes about 20 seconds on one core, checks Lemmas 19 and 20, the error bounds and conditions (i)–(v) in the proof of Proposition 22, and the inequality in the proof of Corollary 11 at $x = 6 \cdot 10^{14}$. The independent implementation mentioned after Proposition 22 works in ball arithmetic with the python-flint library and takes about two seconds. It checks Lemma 19, both rows of Table 1, the error bounds in the proof of Proposition 22 and the inequality in the proof of Corollary 11.

Theorem 6. A second program, which uses the Python standard library and NumPy and takes about two minutes, checks Proposition 42, the Pratt certificate for $4 \cdot 10^{18} - 113$ used in Lemma 34, and all numerical statements of Lemmas 30, 31, 39 and 40 and Proposition 29. It also checks that the charge it evaluates is the expression $\mathfrak{c}$ of Proposition 33 with β_κ as in the proof of Proposition 42. It reads three data files: the table of [14] as distributed by its authors, the 48 families $\mathbf{c}_i$, and the first 1.4×10^6 ordinates of zeros of ζ from [10], truncated to 11 decimals. Run without the zeros, in about three and a

half minutes, it checks Proposition 43 and the value for the rows $h_i \le 2^{64}$ in the remark at the end of Appendix F, together with the constants and slab conditions these use.

Theorem 2 and Proposition 5. The retained certificates of Section 5 and the program that replays them are published at `https://github.com/charlieoneill11/maths-releases`; Section 5.3 says what this replay establishes. The programs of the 2026 computation, its schedule, the record of its audit and a ledger of all ranges are published there too. The witness verifier is a C++ program using GMP, and Pratt certificates are checked by a separate Python program.

H Formal verification in Lean

Theorem 4 has also been proved in the Lean 4 proof assistant [9] with the Mathlib library [11], from the Riemann hypothesis and five published results, which enter as explicit hypotheses. With the computation of Section 5 as a further hypothesis, so have Theorems 1 and 2. The development uses version 4.29.1 of Lean and the matching release of Mathlib, and is available at `https://github.com/charlieoneill11/maths-releases`.

Statements. The formal statements state Theorems 1, 2 and 4 as written. The Riemann hypothesis enters as Mathlib's proposition `RiemannHypothesis`. The formal proof of the Riemann hypothesis cited in [1] uses a later version of Lean and Mathlib, so it is not imported here. Five published results are stated as hypotheses: the computation of [20] as used in Proposition 3; [4, Theorem 5] under RH, as used in Lemma 10; the truncated explicit formula of [7, Ch. 17] quoted in the proof of Lemma 14; the bound of [21, Cor. 1] used in Lemma 15; and [12, Thm. 2]. The two parts of the computation of Section 5 that the proof uses are also stated as hypotheses: a prime in $(m^2, (m+1)^2)$ for $70{,}500{,}000{,}000{,}000 < m \le 10^{14}$, and a prime p with $n^{11} < p^5 < (n+1)^{11}$ for $n_2 < n < 35 \cdot 10^{12}$, where $n_2 = 5{,}336{,}699{,}231{,}206$. Lean proves Theorem 4 from RH and the five published results, Theorem 2 from the published results and the computation, and Theorem 1 from all three. The formal statements were reviewed against the paper and then kept unchanged.

Proof. Every result of Sections 2 to 4, and the proofs of Theorem 1 and Proposition 5, has a Lean counterpart. Proposition 17 is proved for $\mathcal{B} = \emptyset$, the only case used in Section 4. The appendices on Theorem 6 and the primality certificates of Lemma 23 are not formalised. The formal proof follows the paper. The differences are minor: Lemma 15 is proved for $t < e$ by mono-

tonicity, Lemma 16 integrates from height 0 and absorbs the boundary term in the existing error allowance, and the variation integrals of Lemma 19 are bounded by a different splitting that gives the same constants. The formalisation also showed that Lemma 14 needs $ae^{-KD} \geq 2$, since the truncated explicit formula is valid only for $x \geq 2$. Its previous hypotheses, $a > 1$ and $KD < \log a$, did not ensure this. The applications in Proposition 17 satisfy it, since there $ae^{-KD} > e^{59}$, and the statement above has been corrected. The numerical inputs of Section 4 are recomputed in Lean with directed rounding, using logarithm and cosine enclosures proved correct in Lean: the bounds of Lemma 19 for F_{12}, the majorant of Lemma 20, and the entries of Table 1. Lean proves the bounds as printed here.

Trust. Five checkers are evaluated with Lean's `native_decide` tactic, which runs compiled code, so accepting the formal proof means trusting the Lean compiler for these evaluations as well as the Lean kernel. The proofs of Theorems 1 and 4 use these five evaluations, which take about five minutes in all; the proof of Theorem 2 uses none. Apart from these evaluations, the proofs use only the three standard axioms of Lean: propositional extensionality, the axiom of choice and the soundness of quotients. They contain no unproved steps and declare no further axioms.

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