The optimal constant for four-colour quadratic van der Waerden numbers

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Abstract

For integers e, f, not both zero, let $W(ex^2 + fx; 4)$ be the least N such that every four-colouring of $\{1, \dots, N\}$ has two distinct integers of the same colour whose difference is $ed^2 + fd$ for some integer d. Kumthekar, Gasarch, Regli and Whitehouse proved that $W(ex^2 + fx; 4) \leq 64(|e| + |f|) + 1$, and asked for the optimal constant C_4 in a bound of this form, which they showed lies between 57 and 64. We prove that $W(ex^2 + fx; 4) \leq 57|e| + 39|f| + 1$ for all such e, f, so $C_4 = 57$. Both coefficients are attained, by x^2 and by $x^2 + x$. The proof maps the lattice $\mathbb{Z}^2$ to the integers by $(u, v) \mapsto |f|u + |e|v$, which turns the steps $(d, \pm d^2)$ into the differences $\pm(ed^2 \pm fd)$. With the steps for $1 \leq d \leq 7$ the lattice graph is four-colourable by an 8×16 periodic tile, so near the axis $f = 0$ no finite subgraph of it can be an obstruction on its own. There we use the fact that a colouring pulled back from the integers is invariant under the kernel of this map, and a SAT computation with a checked DRAT proof shows that every proper colouring of a 12×58 rectangle contains one of sixteen small configurations, each of which forces a periodic pattern that this invariance rules out. We also determine $W(x^2 + jx; 4)$ for $0 \leq j \leq 8$; the values $82, 81, 75, 89, 73, 53$ for $j = 2, 3, 4, 5, 6, 8$ are new.

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AI use and author contributions

The arguments and the programs in this paper were developed and run by Astra (GPT-6.2 Astra, OpenAI) from Charles O'Neill's instructions. Two of the new threshold values, 82 and 81, were first proposed in those instructions; both were recomputed from the definition. The manuscript was prepared from the project's research records with Claude (Anthropic). All computations were re-run before release, and the programs and certificates are available at https://github.com/charlieoneill11/maths-releases.

Charles O'Neill chose the problem and directed the work. Charles O'Neill has checked the arguments and takes full responsibility for the correctness of the results.

1 Introduction

Let $P \in \mathbb{Z}[x]$ be a nonzero polynomial with $P(0) = 0$ and let $r \geq 2$. The polynomial van der Waerden theorem of Bergelson and Leibman [2] implies, as a special case, that for every r there is an N such that every r-colouring of $[N] = \{1, \dots, N\}$ contains distinct integers u, v of the same colour with $v - u = P(d)$ for some integer d. Following [6], write $W(P; r)$ for the least such N. Put

$$\mathcal{D}(P) = \{|P(d)| : d \in \mathbb{Z},\ P(d) \neq 0\}.$$

Then $W(P; r)$ is the least N for which the graph on $[N]$ in which $u \sim v$ whenever $|u - v| \in \mathcal{D}(P)$ has no proper r-colouring.

For four colours and quadratic polynomials the exact values are small but were found only recently. Canakci, Christenson, Fleischman, Gasarch, McNabb and Smolyak showed with a SAT solver that $W(x^2; 4) = 58$ [4], and Kumthekar, Gasarch, Regli and Whitehouse [6] found nine further exact values, among them $W(x^2 + x; 4) = 97$ and $W(x^2 + 7x; 4) = 49$. They also proved a uniform bound [6, Theorem 4.3]:

$$W(ex^2 + fx; 4) \leq 64(|e| + |f|) + 1 \qquad (e, f \in \mathbb{Z},\ (e, f) \neq (0, 0)). \tag{1}$$

Let C_4 be the smallest constant C such that $W(ex^2 + fx; 4) \leq C(|e| + |f|) + 1$ for all such e, f. Since $W(x^2; 4) = 58$, (1) gives $57 \leq C_4 \leq 64$, and the authors state: "One immediate open problem is to determine the optimal constant C_4" [6, Section 6]. We determine it.

Theorem 1. *For all integers e, f, not both zero,*

$$W(ex^2 + fx; 4) \leq 57|e| + 39|f| + 1.$$

In particular $C_4 = 57$.

Both coefficients are best possible. The coefficient 57 is attained by $W(x^2; 4) = 58$. With the first coefficient fixed at 57, the second cannot be smaller than 39, because $W(x^2 + x; 4) = 97 = 57 + 39 + 1$ [6, Theorem 2.3].

Along the way we determine $W(x^2 + jx; 4)$ for $0 \leq j \leq 8$; [6, Lemma 4.2] had shown that each of these values is at most 97.

Theorem 2. *The values of $W(x^2+jx;4)$ for $0 \le j \le 8$ are as follows.*

j	0	1	2	3	4	5	6	7	8
$W(x^2+jx;4)$	58	97	82	81	75	89	73	49	53
Source	*[4]*	*[6]*	*new*	*new*	*new*	*new*	*new*	*[6]*	*new*

The method. Write $E = |e|$ and $F = |f|$; by symmetry it is enough to treat $P(x) = Ex^2 + Fx$ with $E, F \ge 0$ (Lemma 3). The proof of (1) in [6] labels the point $(u, v) \in \mathbb{Z}^2$ by $Fu + Ev$. A lattice step (d, d^2) changes the label by $P(d)$ and a step $(d, -d^2)$ changes it by $-P(-d)$, so a four-colouring of an interval of integers pulls back to a proper colouring of the lattice graph with these steps, on the points whose labels lie in the interval. Kumthekar et al. checked that the graph on the rectangle $[0, 10] \times [0, 64]$ with the steps $1 \le d \le 8$, which has 715 vertices, is not four-colourable [6, Lemma 4.1]. The labels of this rectangle span $64E + 10F$, which gives (1) except for the finitely many ratios F/E at which a step collapses.

A smaller set of 350 points of the same rectangle is still not four-colourable (Proposition 8), and its labels span only $64E + 10F - 2\min(8F, E + F, 2E)$. This proves Theorem 1 whenever $E < 6F$. Near the axis, when F/E is small, no argument of this kind can reach 57. If $7E > 65F$, every difference $|P(d)|$ with $|d| \ge 8$ exceeds $57(E + F)$, so in an interval of length $57(E + F)$ only the steps with $|d| \le 7$ give edges, and the lattice graph with these steps is four-colourable: an 8×16 periodic tile colours all of $\mathbb{Z}^2$ (Proposition 7). Any proof in this range must therefore use the fact that the pulled-back colouring is constant on the fibres of the labelling map.

That is what Section 4 does. We pass to the sublattice $(2\mathbb{Z}) \times (4\mathbb{Z})$, on which the steps with $d = 2, 4, 6$ act as steps $(d', \pm d'^2)$ with $d' = 1, 2, 3$, and consider the four colourings of one of its two parity classes that are constant on the cosets of a sublattice of index four. Each of them is proper, and if such a colouring appears on three consecutive columns of points with labels in a suitable window, a counting argument with the labels forces it on every later column (Lemma 13). The kernel of the labelling map contains a vector that joins two points with different prescribed colours, so this cannot happen (Lemma 14). A computation then shows that every proper colouring of the even part of a 12×58 rectangle contains, somewhere, one of sixteen configurations of eight or ten points that forces such a pattern on three columns (Lemmas 15 and 16). This gives $W(Ex^2 + Fx; 4) \le 57E + 11F + 1$ for $E \ge 6F > 0$ (Theorem 10).

Computer assistance. Four steps are finite computations: the tile (Proposition 7), which can also be checked by hand; the non-colourability of the 350-point set (Proposition 8); the forcing of the sixteen configurations (Lemma 15); and the non-colourability of the 12×58 rectangle without them (Lemma 16). The two non-colourability statements and the upper bounds in Theorem 2 are refutations of CNF formulas by a SAT solver. Each refutation comes with a DRAT proof checked by drat-trim [7], an independent proof checker, and the formulas were compared with a second, independently written encoder. Section 8 lists the programs, the hashes of the inputs and proofs, and the replay commands.

2 Preliminaries

For a polynomial P and an integer $N \geq 1$ let $G(N, P)$ be the graph on $\{0, 1, \ldots, N-1\}$ in which $u \sim v$ when $|u - v| \in \mathcal{D}(P)$. Translation does not change the graph, so $W(P; 4)$ is the least N for which $G(N, P)$ has no proper four-colouring, and $W(P; 4) - 1$ is the largest N for which $G(N, P)$ has one.

Lemma 3. *Let e, f be integers, not both zero, and put $E = |e|$, $F = |f|$. Then $\mathcal{D}(ex^2 + fx) = \mathcal{D}(Ex^2 + Fx)$. In particular $W(ex^2 + fx; 4) = W(Ex^2 + Fx; 4)$.*

Proof. The polynomials $P(x)$, $-P(x)$ and $P(-x)$ have the same set $\mathcal{D}$, and these operations change the signs of e and f independently. □

Lemma 4 ([6, Proposition 2.1])**.** *For every integer $h \geq 1$, $W(hP; r) = h\big(W(P; r) - 1\big) + 1$.*

Lemma 5. *For every integer $F \geq 1$, $W(Fx; 4) = 4F + 1$.*

Proof. Here $\mathcal{D}(Fx)$ is the set of positive multiples of F. On $\{0, \ldots, 4F - 1\}$ colour u by $\lfloor u/F \rfloor$; two integers at a distance divisible by F lie in different blocks. Among $4F + 1$ consecutive integers some residue class modulo F contains five of them, and they are pairwise adjacent. □

Lattice graphs. For $k \geq 1$ let Γ_k be the graph on $\mathbb{Z}^2$ in which (u, v) is adjacent to $(u + d, v + d^2)$ and to $(u + d, v - d^2)$ for $1 \leq d \leq k$. A step $(d, \sigma d^2)$ with $\sigma = \pm 1$ preserves the parity of $u + v$, so Γ_k is the disjoint union of its restrictions to the even points ($u + v$ even) and the odd points. For a finite set $X \subset \mathbb{Z}^2$ we write $\Gamma_k[X]$ for the induced subgraph.

Lemma 6 (Pullback). *Let $E, F \geq 0$, not both zero, put $P(x) = Ex^2 + Fx$ and $\lambda(u, v) = Fu + Ev$. Let $X \subset \mathbb{Z}^2$ be finite, let $k \geq 1$, and suppose that $F \neq dE$ for $1 \leq d \leq k$. Let $a = \min_X \lambda$ and $b = \max_X \lambda$. If χ is a proper four-colouring of $G(b - a + 1, P)$, then $x \mapsto \chi(\lambda(x) - a)$ is a proper four-colouring of $\Gamma_k[X]$. In particular, if $\Gamma_k[X]$ is not four-colourable then $W(P; 4) \leq b - a + 1$.*

Proof. Let $x = (u, v)$ and $y = (u + d, v + \sigma d^2)$ be adjacent in $\Gamma_k[X]$. Then $\lambda(y) - \lambda(x) = Fd + \sigma Ed^2$, which is $P(d)$ if $\sigma = 1$ and $-P(-d)$ if $\sigma = -1$. It is nonzero: for $\sigma = 1$ because E, F are not both zero, and for $\sigma = -1$ because $F \neq dE$. So $\lambda(x) - a$ and $\lambda(y) - a$ are distinct vertices of $G(b - a + 1, P)$ at a distance in $\mathcal{D}(P)$, and they have different colours. □

3 Two lattice graphs

Proposition 7. *Let T be the 8×16 array*

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	0	0	2	2	2	2	2	2	1	3	3	3	3	1
1	2	2	2	3	1	3	1	3	1	3	1	2	2	1	2	2
2	1	1	1	3	1	3	2	2	0	0	0	0	3	0	3	1
3	0	0	0	0	0	0	0	1	3	2	3	1	3	1	3	0
4	2	2	1	3	3	3	3	1	0	1	0	0	2	2	2	2
5	1	3	1	2	2	1	2	2	2	2	2	3	1	3	1	3
6	0	0	0	0	3	0	3	1	1	1	1	3	1	3	2	2
7	3	2	3	1	3	1	3	0	0	0	0	0	0	0	0	1

with rows indexed by $u \bmod 8$ and columns by $v \bmod 16$. The colouring $(u, v) \mapsto T(u \bmod 8,\ v \bmod 16)$ is a proper four-colouring of Γ_7.

Proof. The colouring is periodic with periods $(8, 0)$ and $(0, 16)$, so it suffices to check the 128 points (u, v) with $0 \leq u < 8$, $0 \leq v < 16$ against their 14 neighbours $(u + d, v \pm d^2)$, $1 \leq d \leq 7$, reduced modulo $(8, 16)$. The residues of d^2 modulo 16 are $1, 4, 9, 0, 9, 4, 1$. Both programs of Section 8 carry out these 1792 comparisons. □

In particular, for every n the graph $\Gamma_7[\{0, \ldots, n - 1\}^2]$ is four-colourable; for $n \leq 65$ this is the graph on the $n \times n$ square with every step $(d, \pm d^2)$ that fits. The tile cannot accommodate $d = 8$, since $(8, \pm 64)$ is a period.

Let $Q = \{0, \ldots, 10\} \times \{0, \ldots, 64\}$, the rectangle of [6, Lemma 4.1], and let

$$S = \{(u, v) \in Q :\ u + v \text{ even},\ v \neq 0 \text{ or } u \geq 8,\ v \neq 64 \text{ or } u \leq 2\}.$$

So S consists of the even points of Q in rows 1 to 63, the points $(8,0)$ and $(10,0)$, and the points $(0,64)$ and $(2,64)$. It has 350 points.

Proposition 8. *The graph $\Gamma_8[S]$, which has 350 vertices and 2362 edges, is not four-colourable.*

Proof. This is a finite computation (Section 8). The CNF formula with four Boolean variables per vertex, clauses saying that each vertex has exactly one colour and that no edge is monochromatic, and two unit clauses naming the colours 0 and 1 at the ends of one edge, has 1400 variables and 11 900 clauses. Kissat 4.0.4 [3] refutes it, and drat-trim [7] verifies the resulting DRAT proof. □

The only edges of $\Gamma_8[S]$ with $d = 8$ are the two edges joining $(8,0)$ to $(0,64)$ and $(10,0)$ to $(2,64)$. The even part of Q (358 vertices, 2436 edges) and all of Q (715 vertices, 4864 edges) contain S, so they are not four-colourable either; the latter is [6, Lemma 4.1].

Corollary 9. *Let $E, F \geq 0$, not both zero, with $F \neq dE$ for $1 \leq d \leq 8$. Then*

$$W(Ex^2 + Fx; 4) - 1 \leq 64E + 10F - 2\min(8F,\ E + F,\ 2E).$$

Proof. Let $\lambda(u,v) = Fu + Ev$. The minimum of λ on row 0 of S is $\lambda(8,0) = 8F$. In a row $v \geq 1$ the least u with $u + v$ even is 0 or 1, so the minimum over rows $v \geq 1$ is attained at $(1,1)$ or $(0,2)$, with value $E + F$ or $2E$. Hence $\min_S \lambda = m := \min(8F, E + F, 2E)$. The map $(u,v) \mapsto (10 - u, 64 - v)$ maps S onto itself and sends λ to $10F + 64E - \lambda$, so $\max_S \lambda = 64E + 10F - m$. Now apply Lemma 6 with $k = 8$ and Proposition 8. □

For $E \leq F$ the minimum is $2E$, for $F \leq E \leq 7F$ it is $E + F$, and for $E \geq 7F$ it is $8F$. So the bound of Corollary 9 is $60E + 10F$, $62E + 8F$ and $64E - 6F$ in these three ranges.

4 The near-axis range

Theorem 10. *Let p, q be integers with $p \geq 6q > 0$. Then $W(px^2 + qx; 4) \leq 57p + 11q + 1$.*

Throughout this section $p \geq 6q > 0$, $P(x) = px^2 + qx$ and $N = 57p + 11q$. Suppose, for a contradiction, that χ is a proper four-colouring of $\{0, 1, \ldots, N\}$ for the distances $\mathcal{D}(P)$. Every step of Γ_k, for every k, has a

nonzero label difference, because $pd^2 - qd = d(pd - q) > 0$ for $d \geq 1$. So, by the proof of Lemma 6, the colouring

$$c(u, v) = \chi(qu + pv) \qquad (0 \leq qu + pv \leq N)$$

gives different colours to any two points that are adjacent in Γ_7 and have labels in $[0, N]$.

A coarse lattice. Fix a base point (u_0, v_0) with

$$0 \leq u_0 \leq 7, \qquad 2 \leq v_0 \leq 19, \qquad u_0 + v_0 \text{ even}, \tag{2}$$

and put $z = qu_0 + pv_0$. There are 72 such base points. To $(x, y) \in \mathbb{Z}^2$ we attach the point $\iota(x, y) = (u_0 + 2x, v_0 + 4y)$ and its label

$$\ell(x, y) = qu_0 + pv_0 + 2qx + 4py = z + 2qx + 4py.$$

We call $(x, y) \sim (x + d, y \pm d^2)$ with $d \in \{1, 2, 3\}$ a *coarse step*. It maps under ι to the step $(2d, \pm(2d)^2)$ of Γ_6, and it preserves the parity of $x + y$. Fix a parity $\epsilon \in \{0, 1\}$ and let $C = \{(x, y) : x + y \equiv \epsilon \pmod 2\}$. Let

$$I = [z - 2p,\ z + 38p + 4q].$$

Since $z - 2p = qu_0 + p(v_0 - 2) \geq 0$ and $z + 38p + 4q = q(u_0 + 4) + p(v_0 + 38) \leq 11q + 57p$, the window I lies in $[0, N]$. Its length is $40p + 4q$. Let $A = \{s \in C : \ell(s) \in I\}$ be the set of *active* coarse points. On A the colouring $c'(s) = c(\iota(s))$ is defined, and it is proper for coarse steps.

Lemma 11. *1. The active points with $0 \leq x \leq 2$ are exactly the 15 points $(x, y) \in C$ with $0 \leq x \leq 2$ and $0 \leq y \leq 9$. We call this set the* block *B.*

2. Every point $(x, y) \in C$ with $-6 \leq x \leq 8$ and $0 \leq y \leq 9$ is active.

Proof. We have $\ell(x, y) - z = 2qx + 4py$. If $0 \leq x \leq 2$ and $y \leq -1$ then $2qx + 4py \leq 4q - 4p < -2p$, since $p > 2q$. If $0 \leq x \leq 2$ and $y \geq 10$ then $2qx + 4py \geq 40p > 38p + 4q$. If $0 \leq y \leq 9$ and $-6 \leq x \leq 8$ then $2qx + 4py \geq -12q \geq -2p$ and $2qx + 4py \leq 16q + 36p \leq 38p + 4q$, since $12q \leq 2p$. This proves both statements; for 1 note that the points with $0 \leq y \leq 9$ and $0 \leq x \leq 2$ satisfy the bounds of 2. □

Prescribed colourings. Let $L = \{(a,b) \in \mathbb{Z}^2 : a+b \text{ even}\}$, so that C is a coset of L, and let $2L = \{(2a,2b) : (a,b) \in L\}$, a subgroup of index 4 in L. Define

$$\varphi(x,y) = \big(x \bmod 2,\ \lfloor (y-x)/2 \rfloor \bmod 2\big) \in (\mathbb{Z}/2)^2.$$

Lemma 12. *For $s, s' \in C$ with $s - s' = (a,b)$,*

$$\varphi(s) - \varphi(s') = \big(a \bmod 2,\ \tfrac{b-a}{2} \bmod 2\big).$$

In particular $\varphi(s) = \varphi(s')$ if and only if $s - s' \in 2L$. A coarse step $(d, \sigma d^2)$ changes φ by $(1,0)$ if $(d,\sigma) \in \{(1,+),(3,-)\}$, by $(1,1)$ if $(d,\sigma) \in \{(1,-),(3,+)\}$, and by $(0,1)$ if $d = 2$.

Proof. Write $s = (x,y)$ and $s' = (x',y')$. Since $s, s' \in C$, the integers $y - x$ and $y' - x'$ have the same parity, so $\lfloor (y-x)/2 \rfloor - \lfloor (y'-x')/2 \rfloor = \big((y-x) - (y'-x')\big)/2 = (b-a)/2$. If a is even and $(b-a)/2$ is even then a, b are even and $a/2 + b/2 = a + (b-a)/2$ is even, so $(a,b) \in 2L$; the converse is similar. The values for the coarse steps follow by substituting $(a,b) = (d, \sigma d^2)$. □

For a bijection $\pi\colon (\mathbb{Z}/2)^2 \to \{0,1,2,3\}$ we call $\pi \circ \varphi$ a *prescribed colouring* of C. By Lemma 12 every coarse step changes φ, so a prescribed colouring is proper for coarse steps.

Lemma 13 (Propagation)**.** *Suppose that $c' = \pi \circ \varphi$ on the block B for some bijection π. Then $c' = \pi \circ \varphi$ at every active point (x,y) with $x \geq 0$.*

Proof. By Lemma 111 the statement holds for $x \leq 2$. Let $x \geq 3$, assume the statement for all active points in columns $0, \dots, x-1$, and let $s = (x,y)$ be active. Fix one of the three values $\gamma \neq \varphi(s)$ and put $\beta = \varphi(s) - \gamma \neq 0$. By Lemma 12 there are $d, d' \in \{1,2,3\}$, given by

β	d	d'	$\ell(t') - \ell(t)$
$(1,0)$	1	3	$40p - 4q$
$(1,1)$	3	1	$40p + 4q$
$(0,1)$	2	2	$32p$

such that the points $t = s - (d, d^2)$ and $t' = s - (d', -d'^2)$ of C both satisfy $\varphi(t) = \varphi(t') = \gamma$. Here $\ell(t) = \ell(s) - 2qd - 4pd^2 < \ell(s)$ and $\ell(t') = \ell(s) - 2qd' + 4pd'^2 > \ell(s)$, and $\ell(t') - \ell(t) = 2q(d-d') + 4p(d^2 + d'^2)$ is the value in the last column, which is at most $40p + 4q$, the length of I. If neither t nor t' were active, then $\ell(t) < z - 2p$ and $\ell(t') > z + 38p + 4q$, so $\ell(t') - \ell(t) > 40p + 4q$. Hence one of them is active. It lies in one of the columns $x-3, x-2, x-1$, so by the induction hypothesis its colour is $\pi(\gamma)$. It is joined to s by a coarse step, so $c'(s) \neq \pi(\gamma)$. As this holds for all three $\gamma \neq \varphi(s)$, $c'(s) = \pi(\varphi(s))$. □

Lemma 14. *For no bijection π does c' agree with $\pi \circ \varphi$ on B.*

Proof. For $(a,b) \in L$ write $(a,b) = \alpha(1,1) + \beta(0,2)$ with $\alpha = a$ and $\beta = (b-a)/2$. Then $\ell(s+(a,b)) - \ell(s) = 2qa + 4pb = \alpha K + \beta M$ with $K = 4p+2q$ and $M = 8p$. Let $g = \gcd(K,M)$ and

$$\kappa = \tfrac{M}{g}(1,1) - \tfrac{K}{g}(0,2) \in L.$$

Then $\ell(s+\kappa) = \ell(s)$ for every s, and the first coordinate of κ is $M/g > 0$. In the coordinates (α,β) the subgroup $2L$ consists of the pairs with α and β both even, and M/g and K/g are coprime, so $\kappa \notin 2L$.

Suppose $c' = \pi \circ \varphi$ on B. Take any $s \in B$. The point $s+\kappa$ lies in C, has the same label as s, so it is active, and its first coordinate is positive. By Lemma 13, $c'(s+\kappa) = \pi(\varphi(s+\kappa))$, and $\varphi(s+\kappa) \neq \varphi(s)$ by Lemma 12. But $c'(s+\kappa) = \chi(\ell(s+\kappa)) = \chi(\ell(s)) = c'(s) = \pi(\varphi(s))$, a contradiction. □

Forcing configurations. A *configuration* is a set U of eight or ten points of the block $\{(x,y) : 0 \le x \le 2,\ 0 \le y \le 9\}$, all of one parity, that meets all four classes of φ. We say that c' *contains* U if c' agrees on U with some prescribed colouring $\pi \circ \varphi$; equivalently, two points of U have the same colour if and only if their difference lies in $2L$.

Lemma 15. *Let $\mathcal{U}$ be the sixteen configurations listed in Table 1. If c' contains some $U \in \mathcal{U}$, agreeing there with $\pi \circ \varphi$, then $c' = \pi \circ \varphi$ on B.*

Proof. By Lemma 112, every point of C in the box $X = \{-6 \le x \le 8,\ 0 \le y \le 9\}$ is active, so c' is a proper colouring of the coarse-step graph on $X \cap C$. Start from the colours $\pi(\varphi(s))$ on U and repeatedly give a point of $X \cap C$ the fourth colour when three of its coarse neighbours in X already carry the other three. Each such step is forced in every proper colouring that agrees with $\pi \circ \varphi$ on U. For each of the sixteen configurations this process colours all 15 points of B, and the colours agree with $\pi \circ \varphi$; up to renaming of the colours the process does not depend on π. This is a finite check, carried out by two independent programs (Section 8). □

The configurations were found by enumerating, for each parity, the inclusion-minimal subsets of the block that force all of it by this process inside $0 \le x \le 8$. There are four of size eight and four of size ten for each parity. The reflections $x \mapsto 2-x$ and $y \mapsto 9-y$ map the block onto itself and permute the classes of φ; the second one exchanges the two parities. Closing under them gives the sixteen configurations of Table 1. A configuration

$\lvert U\rvert$	Points (x, y) of the configuration
8	$(0,0),(0,2),(0,6),(1,3),(1,5),(1,7),(1,9),(2,0)$
8	$(0,0),(0,2),(1,3),(1,5),(1,7),(1,9),(2,0),(2,2)$
8	$(0,0),(0,6),(1,3),(1,5),(1,7),(1,9),(2,0),(2,6)$
8	$(0,0),(1,3),(1,5),(1,7),(1,9),(2,0),(2,2),(2,6)$
8	$(0,3),(0,7),(0,9),(1,0),(1,2),(1,4),(1,6),(2,9)$
8	$(0,3),(0,9),(1,0),(1,2),(1,4),(1,6),(2,3),(2,9)$
8	$(0,7),(0,9),(1,0),(1,2),(1,4),(1,6),(2,7),(2,9)$
8	$(0,9),(1,0),(1,2),(1,4),(1,6),(2,3),(2,7),(2,9)$
10	$(0,0),(0,2),(0,6),(0,8),(1,1),(1,3),(1,5),(1,7),(2,0),(2,8)$
10	$(0,0),(0,2),(0,8),(1,1),(1,3),(1,5),(1,7),(2,0),(2,2),(2,8)$
10	$(0,0),(0,6),(0,8),(1,1),(1,3),(1,5),(1,7),(2,0),(2,6),(2,8)$
10	$(0,0),(0,8),(1,1),(1,3),(1,5),(1,7),(2,0),(2,2),(2,6),(2,8)$
10	$(0,1),(0,3),(0,7),(0,9),(1,2),(1,4),(1,6),(1,8),(2,1),(2,9)$
10	$(0,1),(0,3),(0,9),(1,2),(1,4),(1,6),(1,8),(2,1),(2,3),(2,9)$
10	$(0,1),(0,7),(0,9),(1,2),(1,4),(1,6),(1,8),(2,1),(2,7),(2,9)$
10	$(0,1),(0,9),(1,2),(1,4),(1,6),(1,8),(2,1),(2,3),(2,7),(2,9)$

Table 1: The sixteen forcing configurations, in coarse coordinates. The colours on a configuration are those of a prescribed colouring $\pi \circ \varphi$, for any of the 24 bijections π.

obtained by $x \mapsto 2 - x$ forces the block inside $-6 \le x \le 2$, which is why the box X extends to $x = -6$.

For example, in the first configuration the classes of φ are $\{(0,0)\}$, $\{(0,2),(0,6),(2,0)\}$, $\{(1,5),(1,9)\}$ and $\{(1,3),(1,7)\}$, so the configuration is contained in c' when these four sets receive four different colours, each set in one colour.

Let R be the set of even points of $\{0,\dots,11\} \times \{0,\dots,57\}$, so $|R| = 348$. For a base point (2) and a configuration U, the points $\iota(U)$ lie in R: their coordinates satisfy $u_0 \le u \le u_0 + 4 \le 11$ and $2 \le v \le v_0 + 36 \le 55$, and $u + v \equiv u_0 + v_0 \equiv 0 \pmod 2$.

Lemma 16. *Let ψ be a proper four-colouring of $\Gamma_7[R]$. Then for some base point (2), some $U \in \mathcal{U}$ and some bijection π, the colouring $(x, y) \mapsto \psi(\iota(x, y))$ agrees with $\pi \circ \varphi$ on U.*

Proof. This is a finite computation (Section 8). The CNF formula has four variables for each of the 348 points of R, the clauses saying that each point has exactly one colour, four clauses for each of the 2352 edges of $\Gamma_7[R]$, two unit clauses naming the colours at the ends of one edge, and one clause for each of the $72 \cdot 16 \cdot 24 = 27\,648$ triples (base point, configuration, bijection),

which forbids the corresponding assignment of colours to the points $\iota(U)$. It has 1392 variables and 39 494 clauses. Kissat 4.0.4 refutes it. Drat-trim verified the DRAT proof that Kissat produced, using 35 621 617 resolution steps, and wrote a trimmed proof, which is stored with the programs and which drat-trim also verifies. □

Proof of Theorem 10. The points of R have labels $qu + pv \in [0, 11q + 57p] = [0, N]$, so c restricts to a proper four-colouring of $\Gamma_7[R]$. By Lemma 16 there are a base point, a configuration $U \in \mathcal{U}$ of some parity ϵ, and a bijection π such that $c' = \pi \circ \varphi$ on U, where c' is defined for this base point and parity. By Lemma 15, $c' = \pi \circ \varphi$ on the block B. This contradicts Lemma 14. Hence $\{0, \dots, N\}$ has no proper four-colouring, and $W(P; 4) \le N + 1$. □

5 Proof of Theorem 1

By Lemma 3 we may assume $e = E \ge 0$ and $f = F \ge 0$, not both zero. We show $W(Ex^2 + Fx; 4) - 1 \le 57E + 39F$ in each of five cases.

1. $E = 0$. By Lemma 5, $W - 1 = 4F \le 39F$.
2. $F = 0$. By Lemma 4 and $W(x^2; 4) = 58$ [4], $W - 1 = 57E$.
3. $F = jE$ with $E \ge 1$ and $1 \le j \le 8$. By Lemma 4, $W - 1 = E\big(W(x^2 + jx; 4) - 1\big) \le 96E = 57E + 39E \le 57E + 39F$, using $W(x^2 + jx; 4) \le 97$ [6, Lemma 4.2]; Theorem 2 gives the exact values.
4. $E \ge 6F > 0$. By Theorem 10, $W - 1 \le 57E + 11F$.
5. $0 < E < 6F$ and $F \notin \{E, 2E, \dots, 8E\}$. By Corollary 9, $W - 1 \le 60E + 10F \le 57E + 13F$ if $E \le F$, and $W - 1 \le 62E + 8F \le 57E + 38F$ if $F \le E < 6F$.

These cases cover all pairs: if $E, F > 0$ and $F = jE$ with $1 \le j \le 8$ we are in case 3, and otherwise either $E \ge 6F$ or $E < 6F$ with $F \notin \{E, \dots, 8E\}$. This proves the bound. Since $57E + 39F + 1 \le 57(E + F) + 1$, it gives $C_4 \le 57$, and $W(x^2; 4) = 58$ gives $C_4 \ge 57$. □

The worst case for the second coefficient is $F = E$, where $W - 1 = 96E$; every other case gives at most $57E + 38F$.

6 The thresholds for $x^2 + jx$

Proof of Theorem 2. For each j, Table 3 in Appendix A gives a word of length $W - 1$ over $\{0, 1, 2, 3\}$, where W is the value in Theorem 2. Read as a

colouring of $\{0, \dots, W-2\}$, it gives different colours to any two integers whose difference lies in $\mathcal{D}(x^2+jx)$; so $W(x^2+jx;4) \geq W$. For the upper bound, the CNF formula for proper four-colourings of $G(W, x^2+jx)$, built as in the proof of Proposition 8, is unsatisfiable. For each j a DRAT refutation produced by Glucose [1] through PySAT [5] is stored with the programs and is verified by drat-trim. So $W(x^2+jx;4) \leq W$. □

The graphs $G(W, x^2+jx)$ at the thresholds have 266, 543, 461, 422, 418, 428, 418, 215 and 264 edges for $j = 0, \dots, 8$. The values for $j = 0$, 1 and 7 agree with [4] and [6, Theorem 2.3 and Proposition 2.9].

Remark. For odd $j = 2k+1$ the distances are $|d^2+jd| = 2|T_{d+k} - T_k|$ with $T_s = s(s+1)/2$, so they are all even and the graph splits into its two parity classes. For $j = 1, 3, 7$ the upper bound also follows from a short independent-set search. Dividing the larger class of $\{0, \dots, W-1\}$ by two gives the graph on $\{0, \dots, m-1\}$, with $m = 49, 41, 25$, in which $u \sim v$ when $|u-v| = |T_s - T_k| \neq 0$ for some s. A four-colouring would have a colour class with at least $\lceil m/4 \rceil = 13, 11, 7$ points, and an exhaustive search shows that these graphs have no independent sets of these sizes. For $j = 5$ this argument fails: the set $\{0, 1, 5, 9, 10, 14, 15, 20, 31, 36, 37, 41\}$ is independent in the corresponding graph on 45 points.

7 Remarks

Why the near-axis range needs a different argument. Suppose $7E > 65F > 0$. For $|d| \geq 8$ we have $|P(d)| \geq 64E - 8F > 57(E+F)$, so on the interval $\{0, \dots, 57(E+F)\}$ only the steps with $|d| \leq 7$ give edges. Every such edge lifts to an edge of Γ_7, which is four-colourable by Proposition 7. So in this range a proof of Theorem 1 cannot come from a finite subgraph of the lattice graph alone, as in Corollary 9; it has to use the identifications that the labelling map makes, as Lemma 14 does.

Periodic colourings. Let $M \geq 1$ and let a, b be integers. Theorem 1 implies that no four-colouring of $\mathbb{Z}/M\mathbb{Z}$ gives different colours to x and $x + ad^2 + bd$ for every x and every $1 \leq |d| \leq 7$. Indeed, fix $F \geq 0$ with $F \equiv b \pmod M$. For every $E \equiv a \pmod M$, such a colouring, read modulo M, would colour the integers so that any two at a distance $|Ed^2 + Fd|$ with $1 \leq |d| \leq 7$ differ. When $7E > 65F$ every distance $|Ed^2 + Fd|$ with $|d| \geq 8$ is at least $64E - 8F$, so the colouring would be proper on $\{0, \dots, 64E - 8F - 1\}$ and $W(Ex^2 + Fx; 4) - 1 \geq 64E - 8F$. For large E this contradicts $W - 1 \leq$

$57(E + F)$. Before the proof of Theorem 1 was found, a computer search had excluded such colourings directly for every $M \leq 192$.

8 The computations

The programs, formulas and proofs are available at `https://github.com/charlieoneill11/maths-releases`. Every claim of unsatisfiability is certified by a DRAT proof checked with drat-trim [7], and the formula of Proposition 8 was solved with Kissat 4.0.4 [3]. We re-ran every check on 25 November 2026.

1. *Thresholds and the tile.* One program checks the nine colourings of Table 3 against the distance sets and checks the nine refutations for the upper bounds with drat-trim. It compares the edge sets produced by the encoder with an enumeration that tests each pair through the discriminant of $d^2 + jd \mp k$. It also checks the tile of Proposition 7 on all 128 residues and on the 58×58 square edge by edge, the sizes of Q, its even part and S, and the searches of Section 6. It takes about two seconds.

2. *The forcing formula.* A second program generates the formula of Lemma 16, and a third, written independently of it, audits the formula. The audit rebuilds the edges of $\Gamma_7[R]$ by testing every pair of points and checks the exactly-one clauses and the two unit clauses. For each of the $27\,648$ configuration clauses it finds a base point for which the forbidden assignment forces the block, by the propagation in the proof of Lemma 15, to a prescribed colouring. Drat-trim then checks the proof, in about 40 seconds.

3. *The grids.* Kissat solves the formulas for the even part of Q and for S (Proposition 8), and drat-trim checks the resulting proofs, in a few minutes each.

4. *An independent check.* A further program was written separately for this paper. It takes the words, the tile and the configurations from Table 3, Proposition 7 and Table 1, and checks that they agree with the published data. It enumerates the distance sets directly from $|d^2 + jd|$ and checks the nine words. It rebuilds every CNF formula used here from the definitions, including the $27\,648$ configuration clauses, and compares each, as a set of clauses, with the formula used in the checks above. The only clauses it takes from those formulas

are the two symmetry-breaking unit clauses, and it checks that they name two different colours at the ends of an edge. It runs drat-trim on the stored proofs against its own formulas and re-solves the formula of Proposition 8 with Kissat. It also runs its own propagation for Lemma 15, checks the inequalities of Lemma 11 and the kernel vector of Lemma 14 for all $1 \le q \le 20$ and $6q \le p \le 120$, and checks the case analysis of Section 5 for $0 \le E, F \le 400$. It takes about three minutes.

Object	SHA-256
formula of Lemma 16	`0e39e712bfba5a99cb880ded1c66fcfb` `c175b60b199e48bc432241d31fe28832`
its trimmed proof (binary DRAT)	`1b8b8f21348fd5c06d508b157a49feb5` `881daf203c6ba1c08391e9c3a1ab6829`
the same proof, compressed with xz	`41e5c90b89a72038ab9d551a98752e36` `9b01b6985679e7f1e8934d0738ada6f3`
formula for the even part of Q	`28b969e5f4669bfc7a5a3ac5cab132d3` `47afda1fe96cc456dce92307bccf9a83`
formula of Proposition 8	`4d7a888d0c668c6f11e44ad3af3bf912` `e3d67ea095d2e148d33707c7bc1e4a10`
source code of drat-trim	`d834b649f437e091597f5347f259b9f6` `81087f89ca0844d0cee250a1a1a0c2ee`

Table 2: SHA-256 hashes of the main formulas and proofs.

Acknowledgements

The problem, the bound (1) and the rectangle Q are from the paper of Kumthekar, Gasarch, Regli and Whitehouse [6]. We thank them for posing the question so concretely.

A Colourings at the thresholds

Table 3 gives, for each j, the colours of $0, 1, \ldots, W-2$ in order, where $W = W(x^2 + jx; 4)$; the words are broken into lines of 48 letters.

References

[1] G. Audemard and L. Simon, *Predicting learnt clauses quality in modern SAT solvers*, in: Proc. 21st International Joint Conference on Artificial Intelligence (IJCAI 2009), 399–404.

j	W	colours of $0, 1, \ldots, W-2$
0	58	013210130121301212032120323213232130301303012030120103201
1	97	021332102200330123021332213211330023021132213211330023102300213211330023102100213311021032102100
2	82	010121232020131202313020202313020202313030202313131202313131202323030101313020131
3	81	021120330211203302112033022033021120330211203302112033112033021120330211203302111
4	75	022133300221330001212300011222003112220331122233301212333001113300011132 00
5	89	001312312123001231232300123123230000123123000012312300001232312300123231230012313123120 0
6	73	022211333300022111133300222211133000022211333300022111133300022221113300 0
7	49	020202101010313131232323020202101010313131232323
8	53	000000111111133333322222220000000111111333333222222

Table 3: Proper four-colourings of $G(W-1, x^2+jx)$.

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[4] B. Canakci, H. Christenson, R. Fleischman, W. Gasarch, N. McNabb and D. Smolyak, *On SAT solvers and Ramsey-type numbers*, arXiv:2312.01159 (2023).

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[6] I. Kumthekar, W. Gasarch, W. Regli and I. Whitehouse, *Exact values and bounds for four color quadratic polynomial van der Waerden numbers*, arXiv:2611.14913v1 (2026).

[7] N. Wetzler, M. J. H. Heule and W. A. Hunt, Jr., *DRAT-trim: efficient checking and trimming using expressive clausal proofs*, in: Theory and Applications of Satisfiability Testing (SAT 2014), Lecture Notes in Comput. Sci. **8561**, Springer, 2014, 422–429.