An improved lower bound for the sixth moment of the Riemann zeta function

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Abstract

Let $M_6(T) = \int_0^T |\zeta(\frac{1}{2}+it)|^6 \, \mathrm{d}t$. The conjecture of Conrey and Ghosh predicts $M_6(T) \sim 42\, c_3\, T(\log T)^9$, where $c_3 = a_3/9!$ and a_3 is the usual arithmetic factor. We prove unconditionally that

$$\liminf_{T\to\infty} \frac{M_6(T)}{c_3 T(\log T)^9} \;\geq\; 41.6406029094\ldots,$$

improving the previous record 41.008 of Fraccaro. The argument places every Cauchy–Schwarz test in a common Hilbert space of Schur profiles on the simplex $\{0 < x < y < z < 1\}$, where the target ζ^3 has mass exactly 42. A lower bound then becomes the measure of the region on which physical pairings of test Dirichlet polynomials can be shown to agree with the unitary model. We first determine exactly what the two classical test families can give, namely $264076794201/6400000000 = 41.2619990939\ldots$. We then enlarge the admissible region using the Kuznetsov formula for marked short divisors with the level-averaged spectral large sieves of Deshouillers–Iwaniec and Watt, Heath-Brown's estimate for d_3 in arithmetic progressions, and, for moduli up to $T^{3/5}$, a dual-sum estimate for hyper-Kloosterman sums based on the sums-of-products theorem of Fouvry, Kowalski and Michel, combined with the bounds of Deshouillers and Iwaniec for sums of Kloosterman fractions and with the Kuznetsov formula at the level fixed by the denominators. We also show that enlarging the supports within the classes used here (the dominant shapes, the extreme traces, the shapes with a short coordinate, and the balanced and mixed-row regions, with their reflections) cannot take the projection beyond $41.9462895683\ldots < 42$. The exact masses, this ceiling, the covering argument for the mixed row and the deduction of the bound from the arithmetic pairings have been checked in Lean 4 with Mathlib.

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AI use and author contributions

The mathematical arguments and computer programs in this paper were developed by Astra (GPT-6.2 Astra, OpenAI). Fable (Fable 5.2, Anthropic) worked throughout as a second model: it proposed alternative approaches, reviewed arguments and raised objections. The manuscript was prepared from the project's research records with Claude (Anthropic). All computations were re-run before release, and the programs and certificates are available at `https://github.com/charlieoneill11/maths-releases`.

Open-weight models, Kimi K3.1 Pro (Moonshot AI) and DeepSeek V4.1 Flash (DeepSeek), were able to derive a significant proportion of these results when given guidance. We will report the details separately.

Charles O'Neill proposed the problem because the Riemann hypothesis, now a theorem [19], gives Harper's bound $M_6(T) \ll T(\log T)^9$ [6], which controls the error terms of the Cauchy–Schwarz method at the correct order. His technical contribution was to ask for the test-function method to be set up as an optimisation problem and, if it fell short of 42, for its ceiling to be proved. This led to the exact value in Theorem 1 and to the ceiling in Proposition 3. Max Kirkby set up and ran the experiments with open-weight models. Charles O'Neill has checked the arguments and takes full responsibility for the correctness of the results.

1 Introduction

1.1 The sixth moment

For $k \geq 1$ let $M_{2k}(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^{2k}\,\mathrm{d}t$. Hardy and Littlewood evaluated M_2 and Ingham evaluated M_4; no asymptotic is known for $k \geq 3$. Conrey and Ghosh conjectured [1], and the random-matrix model of Keating and Snaith [2] and the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith [3] independently predict,

$$M_6(T) \sim 42\, c_3\, T(\log T)^9, \qquad c_3 = \frac{a_3}{9!}, \tag{1}$$

where

$$a_3 = \prod_p \Big[(1-p^{-1})^9 \sum_{j\geq 0} \binom{j+2}{2}^2 p^{-j}\Big].$$

Throughout we normalise every coefficient by $c_3 T(\log T)^9$, so the conjectured value is 42.

1.2 Previous bounds

The order of magnitude $T(\log T)^9$ is known from below [4, 5]. Harper's upper bound $M_6(T) \ll T(\log T)^9$ [6] rests on the Riemann hypothesis, which is now a theorem of Alpöge [19], so the order of magnitude is known unconditionally. Ng [7] derived the full asymptotic conditionally on a conjecture for ternary additive divisor sums. For the leading constant, the lower-bound method of Conrey and Ghosh pairs ζ^3 with a test Dirichlet polynomial and applies Cauchy–Schwarz. Durkan and Page [8] obtained 34.4012. Fraccaro [9], using two families of tests, obtained 41.008; an exact rational evaluation of Fraccaro's 250 rows gives $41.0089916904\ldots$, and an expanded 397-row certificate gives $41.0298094152\ldots$ (see the remark at the end of Section 3).

1.3 Results

Theorem 1 (Two families)**.** *Let $\mathcal{F}$ be the two test families of Fraccaro, with smooth length profiles and centred derivatives of arbitrary fixed polynomial degree. The supremum of the leading Cauchy–Schwarz coefficients over $\mathcal{F}$ is*

$$C_* = \frac{264076794201}{6400000000} = 41.26199909390625.$$

In particular $\liminf_{T\to\infty} M_6(T)/(c_3 T(\log T)^9) \geq C_*$, and C_* is an upper bound for the coefficient certified by every finite system of tests from $\mathcal{F}$. Restricted to derivative degree at most four, the supremum is strictly smaller than C_*.

Theorem 2 (Main theorem)**.** *Unconditionally,*

$$\liminf_{T\to\infty} \frac{M_6(T)}{c_3 T(\log T)^9} \;\geq\; C = 41.64060290940640373015\ldots,$$

where

$$C = \frac{1126668325607502584615015940142157374558020 61}{27056964762462501481857964726258382399631 36}.$$

Here $C = \mu(U_7)$ is the mass of the explicit region U_7 of Theorem 79.

Proposition 3 (Limit of the method)**.** *Suppose arithmetic pairings were available for every Schur profile supported on the dominant shapes $x + y < z$, on all shapes of trace below 1 or above 2, on all shapes with $x < 1/4$ or $z > 3/4$, on the balanced and mixed-row regions $\mathcal{R}$ and $\mathcal{F}^+$ of Sections 6 and 8, and on the reflections of all of these. Projection onto that span would still give at most*

$$\frac{13860404849860091}{330432202525696} = 41.946289568378975\ldots < 42.$$

The structure of the argument is the following. In Section 2 we put every test in a common Hilbert space $L^2(\Omega)$ of Schur profiles, in which ζ^3 corresponds to the explicit function v with $9!\|v\|^2 = 42$. This identification is proved test by test, from arithmetic evaluations; it does not assume the unknown norm of ζ^3. A lower bound is then $9!\|P_{\mathcal{H}}v\|^2$, the squared projection onto the span $\mathcal{H}$ of the profiles of admissible tests. When $\mathcal{H}$ contains all profiles supported on an open reflection-invariant set U, the bound is $\mu(U)$ for the measure $d\mu = 9!\, v^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Theorem 1 computes this exactly for the two classical families. Theorem 2 enlarges U by proving new arithmetic pairings, in five steps:

New tests	Input	μ of the region
Short cubic, unequal lengths (Theorem 16)	Fourth moment of Dirichlet L-functions	41.3163069650
Marked short divisors (Theorem 23)	Kuznetsov formula, spectral large sieves	41.5917968750
Balanced row (Theorem 52)	d_3 in progressions [15]	41.5958462794
Factorable row (Theorem 67)	Hyper-Kloosterman sums, sums of products [35]	41.6117132746
Mixed row (Theorem 79)	Kloosterman fractions [16], Kuznetsov at fixed level	41.6406029094

The fifth step, Theorem 79, keeps the same witnesses as Theorem 67 but proves the four-shift row for moduli $q \asymp T^{\kappa}$ with κ up to $\frac{3}{5}$, by averaging over the moduli instead of bounding each one. At each step the new family must also be paired with all earlier ones.

Hypotheses. The proofs of Theorem 1, Theorem 16 and Proposition 3 use neither the Riemann hypothesis nor the generalised Riemann hypothesis. The proofs of Theorem 23, Theorem 52, Theorem 67, Theorem 79 and Theorem 2 use Harper's bound $\int_T^{2T} |\zeta(\frac{1}{2}+it)|^6 \,\mathrm{d}t \ll T(\log T)^9$ [6], through Lemma 31 and Lemma 33. Harper's bound rests on the Riemann hypothesis, which is a theorem of Alpöge [19]; this is the sense in which Theorem 2 is unconditional. The four-shift estimates Propositions 68 and 80 themselves use neither hypothesis: their inputs are the Weil and Deligne bounds, the theorem of Fouvry, Kowalski and Michel, the Kuznetsov formula and the spectral large sieve, and the Kim–Sarnak bound for exceptional eigenvalues [26]; Proposition 80 also uses the bound of Deshouillers and Iwaniec for sums of Kloosterman fractions [16, Theorem 12], with the correction of [28], and Pascadi's estimate for the exceptional spectrum [29]. The generalised Riemann hypothesis for Dirichlet L-functions is not used anywhere.

Section 9 assembles the result, and Section 10 shows that enlarging supports in this way cannot reach 42. Section 11 describes the parts of the proof that have been checked in Lean. The computations quoted in the text are carried out in exact rational arithmetic unless stated otherwise, and the programs are available at `https://github.com/charlieoneill11/maths-releases`.

2 Schur profiles and the projection principle

2.1 The chamber and the target

Throughout, $L = \log T$ and $s = \frac{1}{2} + it$. Let

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : 0 < x < y < z < 1\},$$
$$v(x,y,z) = \tfrac{1}{2}(z-y)(z-x)(y-x), \qquad J(x,y,z) = (1-z, 1-y, 1-x),$$

and let μ be the measure $d\mu = 9!\, v^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ on Ω. We write $k = x + y$ and $r = x + y + z$ (the *trace*), and $\langle f, g\rangle = \int_\Omega fg$ for the inner product of $L^2(\Omega)$. A function f on Ω is *J-symmetric* if $f \circ J = f$. For a measurable $A \subseteq \Omega$ we write $L^2_J(A)$ for the space of J-symmetric $f \in L^2(\Omega)$ vanishing almost everywhere off A.

Lemma 4. *J is a measure-preserving involution of Ω, and $v \circ J = v$. Moreover $\int_\Omega v^2 = 42/9!$, that is, $\mu(\Omega) = 42$.*

Proof. J is affine with linear part $-\mathrm{id}$ composed with the transposition of the first and third coordinates, so $|\det J| = 1$, $J \circ J = \mathrm{id}$, and $0 < 1-z < 1-y < 1-x < 1$ shows $J(\Omega) = \Omega$. The differences $(z-y, z-x, y-x)$ are mapped by J to $(y-x, z-x, z-y)$, so $v \circ J = v$. For the integral, $2v = \Delta(x,y,z) := (z-y)(z-x)(y-x)$ changes at most by a sign under permutations of the coordinates, so $\int_\Omega \Delta^2 = \frac{1}{6}\int_{[0,1]^3} \Delta^2$. Expanding both Vandermonde determinants (Heine's identity),

$$\int_{[0,1]^3} \Delta^2 = 3!\,\det\Big(\int_0^1 t^{i+j}\,\mathrm{d}t\Big)_{i,j=0}^{2} = 6\det\Big(\frac{1}{i+j+1}\Big)_{i,j=0}^{2} = \frac{6}{2160}.$$

Hence $\int_\Omega v^2 = \frac{1}{4}\cdot\frac{1}{6}\cdot\frac{6}{2160} = \frac{1}{8640} = \frac{42}{9!}$. □

2.2 Symmetric functions on the unitary group

Let $N \geq 1$, let U be Haar-distributed on $\mathrm{U}(N)$, and put $\langle F, G\rangle_N = \mathbb{E}\, F(U)\overline{G(U)}$ for class functions F, G. Define $e_i(U)$ by $\det(I + uU) =$

$\sum_{i=0}^{N} e_i(U)u^i$, with $e_i = 0$ for $i \notin [0,N]$, and put $D = \det U$. For a partition λ with $\ell(\lambda) \le N$ let $s_\lambda(U)$ be the Schur polynomial in the eigenvalues of U, and let λ' be the conjugate partition. We use the following standard facts [10, 12].

(S1) The s_λ with $\ell(\lambda) \le N$ are orthonormal for $\langle \cdot,\cdot\rangle_N$.

(S2) (Pieri) $e_m s_\nu = \sum_\lambda s_\lambda$, the sum over $\lambda \supseteq \nu$ with $\ell(\lambda) \le N$ such that λ/ν is a vertical strip of m boxes; equivalently $\lambda'_1 \ge \nu'_1 \ge \lambda'_2 \ge \nu'_2 \ge \cdots$ and $|\lambda| = |\nu| + m$ [10, I.(5.17)].

(S3) (Dual Cauchy) $\prod_{k=1}^{N}(1+u_k)^3 = \sum_\lambda s_\lambda(u) s_{\lambda'}(1,1,1)$, where $s_{\lambda'}(1,1,1) = 0$ unless $\lambda_1 \le 3$, and for $\lambda' = (Z,Y,X)$ Weyl's dimension formula gives $s_{\lambda'}(1,1,1) = d_\lambda := \frac{1}{2}(Z-Y+1)(Z-X+2)(Y-X+1)$ [10, I.(4.3′)].

(S4) (Dual Jacobi–Trudi) $s_\lambda = \det(e_{\lambda'_i - i+j})_{1\le i,j\le \ell(\lambda')}$ [10, I.(3.5)].

(S5) (Duality) $e_i = D\,\overline{e_{N-i}}$ for all i. If $\lambda_1 \le 3$ and $\lambda' = (Z,Y,X)$, then $D^3 \overline{s_\lambda} = s_{\lambda^*}$, where $(\lambda^*)' = (N-X, N-Y, N-Z)$.

(S6) (Stable range) If $m, m' \in \mathbb{Z}^3_{\ge 0}$ and $|m| = |m'| \le N$, then $\langle e_{m_1}e_{m_2}e_{m_3}, e_{m'_1}e_{m'_2}e_{m'_3}\rangle_N$ is the number of 3×3 matrices of non-negative integers with row sums m and column sums m'.

For (S5), if the eigenvalues are $\omega_1,\dots,\omega_N$ then $e_{N-i}(\omega) = \prod_k \omega_k \cdot e_i(\bar\omega) = D\,\overline{e_i(\omega)}$; the second statement is the identity $\det{}^3 \cdot s_\lambda(\omega^{-1}) = s_{(3-\lambda_N,\dots,3-\lambda_1)}(\omega)$, whose conjugate has the stated columns. For (S6), every λ with $|\lambda| \le N$ has $\ell(\lambda) \le N$, so by (S1) the left side is the Hall inner product $\langle e_m, e_{m'}\rangle$ in the ring of symmetric functions; the involution ω is an isometry with $\omega e_m = h_m$, and $\langle h_m, h_{m'}\rangle$ is the stated count [11, §§7.5, 7.9].

The Schur polynomials occurring below all have $\lambda_1 \le 3$. We index them by their column lengths $\lambda' = (Z,Y,X)$, $N \ge Z \ge Y \ge X \ge 0$, and write $\lambda/N = (X/N, Y/N, Z/N) \in \overline{\Omega}$. By (S5), $\lambda^*/N = J(\lambda/N)$.

2.3 Cubic weights and their profiles

For a bounded Borel function F on $[0,1]^3$ put

$$E_N[F] = \sum_{m\in\{0,\dots,N\}^3} F(m/N)\, e_{m_1}e_{m_2}e_{m_3}, \qquad c_\lambda(F) = \langle E_N[F], s_\lambda\rangle_N .$$

Thus $E_N[1] = \det(I+U)^3$. For $(x,y,z) \in \overline{\Omega}$ and $c \in \mathbb{R}^3$ define the *Pieri kernel*

$$\kappa(x,y,z;c) = \big[\min(z,\, c_1+c_2-x) - \max(y,\, c_1+c_2-y,\, c_1,\, c_2)\big]_+$$

if $c \in [0,1]^3$ and $c_1 + c_2 + c_3 = x + y + z$, and $\kappa = 0$ otherwise. The *profile* of F is

$$h_F(x,y,z) = \int_0^1 \int_0^1 F(c)\, \kappa(x,y,z;c)\, \mathrm{d}c_1\, \mathrm{d}c_2, \qquad c_3 = x+y+z-c_1-c_2. \tag{2}$$

For fixed c the kernel is continuous in (x,y,z) except where $c_3 = r - c_1 - c_2 \in \{0,1\}$, a null set of (c_1,c_2) for each (x,y,z), and $0 \le \kappa \le 1$. If F is continuous, dominated convergence shows that h_F is continuous on $\overline{\Omega}$. For a general bounded Borel F the profile is bounded and measurable but need not be continuous: for $F(c) = \mathbf{1}_{c_1+c_2+c_3 \in \mathbb{Q}}$ one gets $h_F = \mathbf{1}_{x+y+z \in \mathbb{Q}}\, v$ (by Lemma 6(ii) below).

Lemma 5 (Pieri multiplicities)**.** *Let $K_\lambda(m)$ be the multiplicity of s_λ in $e_{m_1} e_{m_2} e_{m_3}$, where $\lambda' = (Z,Y,X)$. Then $K_\lambda(m) = 0$ unless $|m| = |\lambda|$ and $0 \le m_3 \le N$, in which case $K_\lambda(m)$ is the number of integers u with*

$$\max(Y,\, m_1 + m_2 - Y,\, m_1,\, m_2) \le u \le \min(Z,\, m_1 + m_2 - X).$$

In particular $|K_\lambda(m) - N\kappa(\lambda/N; m/N)| \le 1$ for all m, and $\sum_m K_\lambda(m) = d_\lambda$.

Proof. By (S2) applied to e_{m_2} and the one-column shape (1^{m_1}), $e_{m_1} e_{m_2}$ is the sum of s_ν over the two-column shapes $\nu' = (u,w)$ with $u \ge m_1 \ge w \ge 0$ and $u + w = m_1 + m_2$, that is, $\max(m_1,m_2) \le u \le m_1 + m_2$ and $w = m_1 + m_2 - u$. Applying (S2) again, s_λ occurs in $e_{m_3} s_\nu$ (once) if and only if $|\lambda| = |\nu| + m_3$, $0 \le m_3 \le N$ and $Z \ge u \ge Y \ge w \ge X$; the conditions $w \le Y$ and $w \ge X$ read $u \ge m_1 + m_2 - Y$ and $u \le m_1 + m_2 - X$. This gives the count. The number of integers in a real interval differs from the positive part of its length by at most one, and in scaled units the interval has length $N\kappa(\lambda/N; m/N)$. Finally $\sum_m K_\lambda(m)$ is the coefficient of s_λ in $(\sum_i e_i)^3 = \det(I+U)^3$, which is d_λ by (S3). □

Lemma 6 (Profiles)**.** (i) *If F is continuous, then $\sup_\lambda |N^{-3} c_\lambda(F) - h_F(\lambda/N)| \to 0$ as $N \to \infty$.*

(ii) *$h_1 = v$. Consequently $|h_F| \le \|F\|_\infty v$ for every bounded F, and $F \mapsto h_F$ is linear.*

(iii) *If F, G are continuous and real-valued, then as $N \to \infty$*

$$N^{-9} \langle E_N[F], E_N[G] \rangle_N \to \langle h_F, h_G \rangle,$$
$$N^{-9} \langle E_N[F], D^3 \overline{E_N[G]} \rangle_N \to \langle h_F, h_G \circ J \rangle.$$

Proof. (i) By Lemma 5, $N^{-3}c_\lambda(F) = N^{-2}\sum_{m_1,m_2} F(m/N)\kappa(\lambda/N; m/N) + O(N^{-1})$, the sum over integers $0 \le m_1, m_2 \le N$ with $m_3 = |\lambda| - m_1 - m_2$, since only $O(N^2)$ terms are non-zero and each differs from its replacement by $O(N^{-3})$. The main term is a Riemann sum, with mesh $1/N$, for the integral (2) at $(x, y, z) = \lambda/N$. Its integrand is bounded by $\|F\|_\infty$; off the two lines $c_3 \in \{0, 1\}$ it has a modulus of continuity bounded by that of F plus $2|\cdot|$, uniformly in (x, y, z), because κ is a positive part of a difference of a minimum and a maximum of affine functions with coefficients in $\{0, \pm 1\}$. The $O(N)$ cells of the Riemann sum that meet the two lines contribute $O(N^{-1})$. Hence the error tends to 0 uniformly in λ.

(ii) Apply (i) with $F = 1$. Since $\sum_m K_\lambda(m) = d_\lambda$ and $N^{-3}d_\lambda = v(\lambda/N) + O(N^{-1})$, we get $h_1(\lambda/N) = v(\lambda/N) + o(1)$ uniformly. Both functions are continuous on $\overline{\Omega}$ and the points λ/N become dense, so $h_1 = v$. As $\kappa \ge 0$, $|h_F| \le \|F\|_\infty h_1$. Linearity is clear.

(iii) By (S1), $\langle E_N[F], E_N[G]\rangle_N = \sum_\lambda c_\lambda(F)c_\lambda(G)$, the sum over $\lambda' = (Z, Y, X)$ with $N \ge Z \ge Y \ge X \ge 0$. There are $O(N^3)$ such λ, and by (i) and (ii) both $N^{-3}c_\lambda(\cdot)$ are bounded and equal $h_\cdot(\lambda/N) + o(1)$ uniformly. Hence $N^{-9}\sum_\lambda c_\lambda(F)c_\lambda(G) = N^{-3}\sum_\lambda h_F(\lambda/N)h_G(\lambda/N) + o(1)$, a Riemann sum for the continuous function $h_F h_G$ on $\overline{\Omega}$. For the second limit, (S5) gives $D^3\overline{E_N[G]} = \sum_\lambda c_\lambda(G)s_{\lambda^*}$, so the inner product is $\sum_\lambda c_{\lambda^*}(F)c_\lambda(G)$, and $\lambda^*/N = J(\lambda/N)$. The same argument gives $\int_\Omega h_F(J\cdot)h_G = \langle h_F, h_G \circ J\rangle$ by Lemma 4. For complex-valued continuous F, G the same proof applies with $c_\lambda(G)$ replaced by $\overline{c_\lambda(G)}$, and the limits are $\int_\Omega h_F\overline{h_G}$ and $\int_\Omega h_F\,\overline{h_G \circ J}$. □

Remark. In particular $N^{-9}\|\det(I + U)^3\|_N^2 \to \|v\|^2 = 42/9!$. This is only the motivation for the normalisation: the unitary model enters the proofs below solely through Lemma 6, which computes inner products of explicit profiles.

2.4 Faithful families and the projection principle

A *test* is a family $B = (B_T)_{T \ge T_0}$ of continuous functions $B_T : \mathbb{R} \to \mathbb{C}$. Let $\mathcal{W}$ be the set of non-negative $\Phi \in C_c^\infty((1, 2))$, $\Phi \not\equiv 0$. For $\Phi \in \mathcal{W}$ and tests B, B' put

$$\mathcal{N}_T = a_3\, TL^9 \int \Phi, \qquad \mathrm{N}_T(B) = \int_{\mathbb{R}} \Phi(t/T)\,\zeta(s)^3\,\overline{B_T(t)}\,\mathrm{d}t,$$
$$\mathrm{G}_T(B, B') = \int_{\mathbb{R}} \Phi(t/T)\,B_T(t)\overline{B'_T(t)}\,\mathrm{d}t.$$

Definition 7. A *family* is a set $\mathcal{T}$ of pairs (B, h_B), where B is a test and $h_B \in L^2(\Omega)$ is real-valued (the *profile* of B). It is *faithful* if for every $\Phi \in \mathcal{W}$ and all $(B, h_B), (B', h_{B'}) \in \mathcal{T}$,

$$\frac{\mathrm{N}_T(B)}{\mathcal{N}_T} \to \langle v, h_B \rangle, \qquad \frac{\mathrm{G}_T(B, B')}{\mathcal{N}_T} \to \langle h_B, h_{B'} \rangle \qquad (T \to \infty).$$

Its *span* $\mathcal{H}(\mathcal{T})$ is the closed linear span of $\{h_B\}$ in $L^2(\Omega)$.

If $\mathcal{T}$ is faithful, so is the family of all finite real linear combinations $(\sum c_i B_i, \sum c_i h_{B_i})$, by bilinearity. The union of two faithful families is faithful if and only if all *cross-pairings* $\mathrm{G}_T(B, B')$, with B from one family and B' from the other, have the limits $\langle h_B, h_{B'} \rangle$. This is why, in Sections 5 to 7, each new family is paired with every earlier one before its support is added.

Proposition 8 (Projection principle)**.** *Let $\mathcal{T}$ be faithful and $\mathcal{H} = \mathcal{H}(\mathcal{T})$. Then*

$$\liminf_{T \to \infty} \frac{M_6(T)}{c_3 T L^9} \;\geq\; 9! \, \|P_{\mathcal{H}} v\|^2,$$

where $P_{\mathcal{H}}$ is orthogonal projection. If $\mathcal{H} \supseteq L^2_J(U)$ for a J-invariant measurable set $U \subseteq \Omega$, then $9! \|P_{\mathcal{H}} v\|^2 \geq \mu(U)$.

Proof. All profiles and all coefficients of combinations here are real. Put $C = 9! \|P_{\mathcal{H}} v\|^2$. If $C = 0$ there is nothing to prove, so assume $C > 0$. Let $\epsilon > 0$. Since $\langle v, h \rangle = \langle P_{\mathcal{H}} v, h \rangle$ for $h \in \mathcal{H}$, and the map $h \mapsto \langle v, h \rangle^2 / \|h\|^2$ is continuous at $h = P_{\mathcal{H}} v \neq 0$ with value $\|P_{\mathcal{H}} v\|^2$, there is a finite real combination $h = \sum_i c_i h_{B_i}$ of profiles of $\mathcal{T}$ with $h \neq 0$ and $9! \langle v, h \rangle^2 / \|h\|^2 > C - \epsilon$. Put $B = \sum_i c_i B_i$. By Cauchy–Schwarz with the weight $\Phi(t/T) \geq 0$,

$$|\mathrm{N}_T(B)|^2 \leq \int \Phi(t/T) |\zeta(s)|^6 \, \mathrm{d}t \cdot \mathrm{G}_T(B, B).$$

By faithfulness $\mathrm{N}_T(B)/\mathcal{N}_T \to \langle v, h \rangle$ and $\mathrm{G}_T(B, B)/\mathcal{N}_T \to \|h\|^2 > 0$. As $\mathcal{N}_T = 9! \, c_3 T L^9 \int \Phi$, we obtain, for $T \geq T_1(\epsilon, \Phi)$,

$$\int \Phi(t/T) |\zeta(\tfrac{1}{2} + it)|^6 \, \mathrm{d}t \;\geq\; (C - 2\epsilon) \, c_3 \, T L^9 \int \Phi. \tag{3}$$

Now fix an integer $K \geq 1$ and choose $\Phi \in \mathcal{W}$ with $\Phi \leq \mathbf{1}_{(1,2)}$ and $\int \Phi \geq 1 - \epsilon$. For T large, applying (3) at the K heights $T_k = 2^{-k} T$,

$$M_6(T) \;\geq\; \sum_{k=1}^{K} \int_{T_k}^{2T_k} |\zeta(\tfrac{1}{2} + it)|^6 \, \mathrm{d}t \;\geq\; (C - 2\epsilon)(1 - \epsilon) \, c_3 \sum_{k=1}^{K} 2^{-k} T (\log T_k)^9.$$

Since $\log T_k/\log T \to 1$ for each fixed k, the $\liminf$ of $M_6(T)/(c_3 T L^9)$ is at least $(C-2\epsilon)(1-\epsilon)(1-2^{-K})$. Letting $\epsilon \to 0$ and $K \to \infty$ proves the first claim.

For the second, if $\mu(U) = 0$ the claim follows from $\|P_{\mathcal{H}} v\|^2 \geq 0$, so assume $\mu(U) > 0$. Then $v\mathbf{1}_U$ is J-symmetric because $v \circ J = v$ and $J(U) = U$, so $v\mathbf{1}_U \in \mathcal{H}$ and $\|P_{\mathcal{H}} v\| \geq \langle v, v\mathbf{1}_U\rangle / \|v\mathbf{1}_U\| = \|v\mathbf{1}_U\|$. Thus $9!\|P_{\mathcal{H}} v\|^2 \geq 9! \int_U v^2 = \mu(U)$. □

No step uses the size of $\int |\zeta|^6$: the profile of each test is identified from its own arithmetic evaluation, and the target enters only through the numerators $\mathrm{N}_T(B)$. Every finite combination is fixed before $T \to \infty$.

3 The two classical families: proof of Theorem 1

3.1 The families

For a real function f and a real polynomial p put

$$A_f(s) = \sum_{n \geq 1} f\Big(\frac{\log n}{L}\Big) n^{-s}, \qquad C_p(s) = p\Big(\frac{1}{2} + \frac{1}{L}\frac{d}{ds}\Big)\zeta(s),$$

$$\Pi H = \tfrac{1}{2}\big(H + \chi(s)^3 \overline{H}\big),$$

where $\zeta(s) = \chi(s)\zeta(1-s)$. For $s = \frac{1}{2} + it$, $\overline{\zeta(s)^3}\chi(s)^3 = \zeta(s)^3$, so $\mathrm{N}_T(\Pi H) = \mathrm{Re}\,\mathrm{N}_T(H)$.

U-tests. Data (a_0, b_0, f, g, p) with $0 < a_0 \leq b_0$, $3a_0 + 5b_0 < 2$, real $f \in C_c^\infty((0, a_0))$, $g \in C_c^\infty((0, b_0))$ and a real polynomial p. The test is ΠU with $U = \chi(s) C_p(s) A_f(s) A_g(\bar{s})$, its cubic weight is $F_U(c) = f(c_1)\, g(1-c_2)\, p(\frac{1}{2} - c_3)$, and its profile is $\frac{1}{2}(h_{F_U} + h_{F_U} \circ J)$.

V-tests. Data (η, f, g, p) with $0 < \eta < \frac{1}{4}$, real polynomials f, g on $[0, \eta]$ vanishing at 0 and η and extended by 0, and a real polynomial p. The test is ΠV with $V = C_p(s) A_f(s) A_g(s)$, its cubic weight is $F_V(c) = f(c_1) g(c_2) p(\frac{1}{2} - c_3)$, and its profile is $\frac{1}{2}(h_{F_V} + h_{F_V} \circ J)$.

Let $\mathcal{T}_2$ be the family of all these pairs. The cubic weights record the matrix analogues: by (S5), $E_N[F_U] = \sum_{i,A,B} p(\frac{1}{2} - \frac{i}{N}) f(\frac{A}{N}) g(\frac{B}{N})\, e_i\, e_A\, D\overline{e_B}$ mirrors U, with D in the role of χ, and Π corresponds to $h \mapsto \frac{1}{2}(h + D^3\overline{h})$, which acts on profiles by J (Lemma 6(iii)).

For $0 \leq \alpha, \beta \leq 1$ and a polynomial p let

$$H_{\alpha,\beta,p}(x, y, z) = p(\tfrac{1}{2} - j)\, \kappa(x, y, z; \alpha, \beta, j), \qquad j = x + y + z - \alpha - \beta,$$

be the *point feature*. By Fubini, the raw profiles are

$$h_{F_U} = \iint f(\alpha)g(\beta)H_{\alpha,1-\beta,p}\,\mathrm{d}\alpha\,\mathrm{d}\beta, \qquad h_{F_V} = \iint f(\alpha)g(\beta)H_{\alpha,\beta,p}\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

3.2 The arithmetic input

We define three kernels. Let $X = (X_0, \dots, X_3)$ be uniformly distributed on the simplex $\{X \in [0,1]^4 : \sum X_i = 1\}$, and for $\varrho \in \mathbb{R}^4$ put $\Delta(\varrho) = \prod_{0\le i<k\le 3}(\varrho_i - \varrho_k)$. For $\alpha, \beta \in [0,1]$ put $\varrho_w = (1, \alpha+\beta-w, w, 0)$ and

$$\mathcal{K}_p(\alpha,\beta) = \int_{\max(0,\alpha+\beta-1)}^{\min(\alpha,\beta)} \frac{\Delta(\varrho_w)}{12}\,\mathbb{E}\,p\Big(\sum_{i=0}^{3}(\varrho_{w,i} - \tfrac12)X_i\Big)\,\mathrm{d}w. \tag{4}$$

For $a,b,c,d \ge 0$ with $a+b+c+d<1$ and polynomials p, q let

$$\mathcal{B}_{p,q}(a,b;c,d) = \int_{\mathcal{R}_0}\int_{-w}^{w} p(\delta-h)\,q(-\delta-h)\,\mathrm{d}h\,\mathrm{d}\varrho, \tag{5}$$

where $\mathcal{R}_0 = \{\varrho \in [0,\infty)^{2\times 2} : \varrho_{11}+\varrho_{12} \le a,\ \varrho_{21}+\varrho_{22} \le b,\ \varrho_{11}+\varrho_{21} \le c,\ \varrho_{12}+\varrho_{22} \le d\}$, $R = a+b-|\varrho|$, $S = c+d-|\varrho|$, $w = \frac12(1-R-S)$ and $\delta = \frac12(R-S)$. For $c \in [0,\infty)^3$ and $0 \le \sigma \le |c| < 1$ let

$$\mathcal{M}_{p,q}(c;\sigma) = \int_{\Sigma}\int_{\prod_k[-\varrho_k/2,\varrho_k/2]} p\Big(\tau - \sum_k y_k\Big)\,q\Big(\tau + \sum_k y_k\Big)\,\mathrm{d}y\,\mathrm{d}a_1\,\mathrm{d}a_2, \tag{6}$$

where $\Sigma = \{a \in [0,\infty)^3 : a_1+a_2+a_3 = \sigma,\ a_k \le c_k\}$, $\varrho_k = c_k - a_k$, and $\tau = \frac12(1-|c|+\sigma)$; put $\mathcal{M}_{p,q}(c;\sigma) = 0$ if $\sigma > |c|$. Write $p^-(X) = p(-X)$.

Theorem 9 (Fraccaro). *Let $\Phi \in \mathcal{W}$ and let ΠU_i, ΠV_i be tests from $\mathcal{T}_2$ with data (f_i, g_i, p_i), $i = 1,2$. For a kernel $\mathcal{Q}(a,b,c,d)$ write $[\mathcal{Q}] = \iiiint f_1(a)g_1(b)f_2(c)g_2(d)\,\mathcal{Q}(a,b,c,d)$. Then, as $T \to \infty$,*

$$\begin{aligned}
\mathrm{N}_T(\Pi U_1)/\mathcal{N}_T &\to \iint f_1(a)g_1(b)\,\mathcal{K}_{p_1}(a,1-b)\,\mathrm{d}a\,\mathrm{d}b,\\
\mathrm{N}_T(\Pi V_1)/\mathcal{N}_T &\to \iint f_1(a)g_1(b)\,\mathcal{K}_{p_1}(a,b)\,\mathrm{d}a\,\mathrm{d}b,\\
\mathrm{G}_T(\Pi U_1,\Pi U_2)/\mathcal{N}_T &\to \tfrac12\big[\mathcal{B}_{p_1,p_2}(a,d;b,c) + \mathcal{B}_{p_1,p_2^-}(a,c;b,d)\big],\\
\mathrm{G}_T(\Pi V_1,\Pi V_2)/\mathcal{N}_T &\to \tfrac12\big[\mathcal{B}_{p_1,p_2}(a,b;c,d)\big],\\
\mathrm{G}_T(\Pi U_1,\Pi V_2)/\mathcal{N}_T &\to \tfrac12\big[\mathcal{M}_{p_1,p_2^-}(b,c,d;a) + \mathcal{M}_{p_1^-,p_2^-}(a,c,d;b)\big].
\end{aligned}$$

For $\deg p_i \le 4$ this is [9, Props. 3 and 7–10, and (2.65)]; the arbitrary fixed degree is Lemma 10 below.

Lemma 10 (Fixed derivative degree)**.** *Theorem 9 holds for real polynomials p_1, p_2 of every fixed degree.*

Proof. Fraccaro's proofs evaluate the shifted mixed moments with C_p replaced by $\zeta(s+\gamma)$, γ in a fixed small complex disc, and obtain the polynomial weights by differentiating in γ. The asymptotic formulae for the shifted U and V numerators at the end of [9, §§2.2.3–2.2.4] and for both diagonal Gram calculations in the proofs of [9, Props. 7–8] hold uniformly for γ in a fixed disc of radius $\asymp 1/L$, so the Cauchy integral formula on that disc gives every fixed number of derivatives with the same leading forms. In the reflection step, [9, Lemma 6] holds at each fixed order; the sum in [9, (2.64)] then runs up to $\deg p_2$, its squared weighted norm is $O(TL^7)$ by the balanced estimate, and its cross term is $O(TL^8)$. The discarded reflected VV estimate also holds uniformly on the disc before differentiation. The implied constants depend on the tests, which are fixed. □

3.3 Identification with Schur profiles

Lemma 11 (Bridge)**.** *For all tests in $\mathcal{T}_2$ the limits in Theorem 9 equal $\langle v, h_B\rangle$ and $\langle h_B, h_{B'}\rangle$, where h_B is the profile of B. Consequently $\mathcal{T}_2$ is faithful.*

Proof. Numerators. Since $v\circ J = v$ and J preserves measure, $\langle v, \frac{1}{2}(h+h\circ J)\rangle = \langle v, h\rangle$. By Fubini it suffices to show $\langle v, H_{\alpha,\beta,p}\rangle = \mathcal{K}_p(\alpha,\beta)$. In (2) write u for the variable of the interval defining κ, and $w = \alpha+\beta-u$. Then u lies in that interval if and only if

$$1 \ge z \ge u \ge y \ge w \ge x \ge 0, \qquad \max(0, \alpha+\beta-1) \le w \le \min(\alpha,\beta),$$

and on this set $j = x+y+z-u-w \in [0,1]$ automatically. Hence

$$\langle v, H_{\alpha,\beta,p}\rangle = \int_{\max(0,\alpha+\beta-1)}^{\min(\alpha,\beta)} \int_{I(\varrho_w)} v\; p\big(\tfrac{1}{2}-x-y-z+\alpha+\beta\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w,$$

where $I(\varrho) = \{\varrho_0 > z > \varrho_1 > y > \varrho_2 > x > \varrho_3\}$ is the interlacing region of $\varrho_w = (1, \alpha+\beta-w, w, 0)$. For $\varrho_0 > \cdots > \varrho_3$ the Dixon–Anderson change of variables (see [13], [14, §4.2]; we include the short proof)

$$X_i = \frac{\prod_{\ell\in\{x,y,z\}}(\varrho_i - \ell)}{\prod_{k\ne i}(\varrho_i - \varrho_k)} \qquad (0 \le i \le 3)$$

is a bijection from $I(\varrho)$ onto the open simplex $\{X_i > 0,\ \sum_i X_i = 1\}$, with $\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3 = \Delta(z,y,x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z/\Delta(\varrho)$, where $\Delta(z,y,x) = 2v$ and $\Delta(\varrho) =$

$\prod_{i<k}(\varrho_i-\varrho_k)$. Indeed, by partial fractions $\prod_\ell(t-\ell)/\prod_i(t-\varrho_i)=\sum_i X_i/(t-\varrho_i)$. Exactly i of the ℓ and i of the ϱ_k exceed ϱ_i, so interlacing makes every X_i positive; comparing the coefficients of t^{-1} and t^{-2} at infinity gives $\sum X_i=1$ and $\sum \varrho_i X_i=\sum\varrho_i-(x+y+z)$. Conversely, given X in the open simplex, $R(t)=\sum_i X_i/(t-\varrho_i)$ is strictly decreasing on each interval $(\varrho_{i+1},\varrho_i)$, from $+\infty$ to $-\infty$, so it has exactly one zero in each of the three intervals; and $R(t)\prod_i(t-\varrho_i)$ is a monic cubic, since $\sum X_i=1$. So it is $\prod_\ell(t-\ell)$ for a unique interlacing triple, and taking residues at ϱ_i recovers X. For the Jacobian, $\partial X_i/\partial\ell=-X_i/(\varrho_i-\ell)$, so by Cauchy's determinant

$$\Big|\det\Big(\frac{\partial X_i}{\partial\ell}\Big)_{i\in\{1,2,3\},\ \ell\in\{z,y,x\}}\Big|=\prod_{i=1}^{3}X_i\cdot\frac{\Delta(\varrho_1,\varrho_2,\varrho_3)\,\Delta(z,y,x)}{\prod_{i=1}^{3}\prod_\ell|\varrho_i-\ell|}=\frac{\Delta(z,y,x)}{\Delta(\varrho)},$$

after inserting the definition of X_i and using $\prod_{i=1}^3\prod_{k\neq i}|\varrho_i-\varrho_k|=\Delta(\varrho_1,\varrho_2,\varrho_3)^2\prod_{i=1}^3(\varrho_0-\varrho_i)$. Since $\sum_i\varrho_{w,i}=1+\alpha+\beta$, the argument of p equals $\sum_i(\varrho_{w,i}-\frac12)X_i$, and the inner integral is $\frac12\Delta(\varrho_w)\int p(\cdot)\,\mathrm{d}X=\frac{\Delta(\varrho_w)}{12}\mathbb{E}\,p(\cdot)$, the simplex having volume $1/6$. This is (4).

Gram entries. By Lemma 6(iii) and $\langle\frac12(h+h\circ J),\frac12(h'+h'\circ J)\rangle=\frac12\langle h,h'\rangle+\frac12\langle h,h'\circ J\rangle$,

$$\langle h_B,h_{B'}\rangle=\lim_{N\to\infty}\frac{1}{2N^9}\Big(\langle E_N[F],E_N[F']\rangle_N+\langle E_N[F],D^3\overline{E_N[F']}\rangle_N\Big)\qquad(7)$$

for the cubic weights F,F' of B,B'. We expand each inner product over the short indices and compute the inner sum over the two derivative indices. Throughout, A,B,C,D denote short indices, scaled as $a=A/N$ and so on; for both families the short indices satisfy $a+b+c+d<1$ (for U-tests $a_0+b_0\le\frac14(3a_0+5b_0)<\frac12$).

(VV). Here $\langle E_N[F],E_N[F']\rangle_N=\sum f_1(a)g_1(b)f_2(c)g_2(d)\,\Sigma_N$ with

$$\Sigma_N=\sum_{i,j}p(\tfrac12-\tfrac{i}{N})q(\tfrac12-\tfrac{j}{N})\,\langle e_ie_Ae_B,e_je_Ce_D\rangle_N,$$

$p=p_1$, $q=p_2$. The inner product vanishes unless $i+A+B=j+C+D$. If $i+j\le N$ the common degree is $\frac12(i+j+A+B+C+D)<N$, so by (S6) it counts 3×3 tables with row sums (i,A,B) and column sums (j,C,D). Such a table is determined by its short-to-short block $(\varrho_{kl})_{k,l\le2}$, whose row sums are at most A,B and column sums at most C,D, together with its corner entry $\ell\ge0$; then $i=\ell+(C+D-|\varrho|)$ and $j=\ell+(A+B-|\varrho|)$. If $i+j>N$, (S5) gives $\langle e_ie_Ae_B,e_je_Ce_D\rangle_N=\langle e_{N-j}e_Ae_B,e_{N-i}e_Ce_D\rangle_N$, and $(N-j)+(N-i)<N$, so the same parametrisation applies to $(N-j,N-i)$.

Summing over the two halves and scaling ($\varrho = N\varrho^{\mathrm{sc}}$, $\ell = N(w+h)$ in the first half and $\ell = N(w-h)$ in the second) gives

$$\Sigma_N = N^5\,\mathcal{B}_{p,q}(a,b;c,d) + O(N^4)$$

uniformly in the short indices: both halves are sums of a fixed polynomial over the lattice points of a five-dimensional polytope of bounded complexity, and in the scaled variables the two derivative weights become $p(\delta-h)q(-\delta-h)$ with $h \in [-w,0]$ and $h \in [0,w]$ respectively. The reflected term vanishes identically: by (S5) every Schur constituent of $D^3\overline{e_j e_C e_D}$ has degree $3N-(j+C+D)$, and $i+A+B = 3N-(j+C+D)$ is impossible because $i,j \le N$ and $A+B+C+D < N$. Summing over the $O(N^4)$ short indices, (7) equals the stated VV limit.

(UU). With $e_{N-B} = D\overline{e_B}$ and $|D| = 1$,

$$\langle e_i e_A e_{N-B}, e_j e_C e_{N-D}\rangle_N = \langle e_i e_A e_D, e_j e_C e_B\rangle_N,$$
$$\langle e_i e_A e_{N-B}, D^3\overline{e_j e_C e_{N-D}}\rangle_N = \langle e_i e_A e_C, e_{N-j} e_B e_D\rangle_N,$$

the second because $D\cdot\overline{D}^3\cdot D = \overline{D}$ and $\overline{D}e_j = \overline{e_{N-j}}$. In the second, substituting $j' = N-j$ turns the weight $q(\frac{1}{2}-\frac{j}{N})$ into $q^-(\frac{1}{2}-\frac{j'}{N})$. Both are now of the form treated in (VV), which gives $N^5\mathcal{B}_{p,q}(a,d;c,b)$ and $N^5\mathcal{B}_{p,q^-}(a,c;b,d)$ up to $O(N^4)$; and $\mathcal{B}$ is symmetric in its last two arguments.

(UV). Similarly

$$\langle e_i e_A e_{N-B}, e_j e_C e_D\rangle_N = \langle e_i e_{N-j} e_A, e_B e_C e_D\rangle_N,$$
$$\langle e_i e_A e_{N-B}, D^3\overline{e_j e_C e_D}\rangle_N = \langle e_A e_C e_D, e_{N-i} e_{N-j} e_B\rangle_N.$$

Consider in general $\langle e_i e_j e_S, e_{C_1} e_{C_2} e_{C_3}\rangle_N$ with $C_1+C_2+C_3 < N$, weighted by $P(\frac{1}{2}-\frac{i}{N})Q(\frac{1}{2}-\frac{j}{N})$. A table with rows (i,j,S) and columns (C_1,C_2,C_3) is determined by the split (a_1,a_2,a_3) of the row S, with $a_k \le C_k$, and by the entries $v_k \in [0, C_k-a_k]$ of the row i; then $j = H - \sum v_k$ with $H = |C| - S$. With $v_k = \frac{1}{2}(C_k-a_k) + Ny_k$ the weights become $P(\tau - \sum y_k)Q(\tau+\sum y_k)$ in scaled variables, so the weighted sum is $N^5\mathcal{M}_{P,Q}(c;\sigma) + O(N^4)$, $\sigma = S/N$. In the first identity substitute $j' = N-j$, so $(P,Q) = (p,q^-)$ with columns (b,c,d) and $\sigma = a$. In the second substitute $i' = N-i$, $j' = N-j$, so $(P,Q) = (p^-,q^-)$ with columns (a,c,d) and $\sigma = b$. This gives the stated UV limit.

Combining with Theorem 9, the limits of $\mathrm{N}_T/\mathcal{N}_T$ and $\mathrm{G}_T/\mathcal{N}_T$ are $\langle v, h_B\rangle$ and $\langle h_B, h_{B'}\rangle$, so $\mathcal{T}_2$ is faithful. □

Remark. The identity of Lemma 11, including the overall normalisation, has also been compared numerically with the exact integer evaluation of Fraccaro's kernels. We took twelve point tests from Fraccaro's certificate: its first and last rows and ten rows drawn at random. It integrates their projected point features over Ω by the midpoint rule on the cubes of side $1/120$ and $1/240$ whose centres lie in Ω, and combines the two by Richardson extrapolation. The largest relative error of a numerator, $|\text{computed} - \text{exact}|/|\text{exact}|$, is $1.5 \cdot 10^{-3}$. The largest Gram discrepancy, divided by the largest absolute Gram entry of the sample, is $5.5 \cdot 10^{-4}$. This is a floating-point consistency check; the proof is the one given above.

3.4 Supports

Put $\varrho = 1 - z$ and define the open sets

$$\begin{aligned} \mathcal{C} &= \{3\min(x,\varrho) + 5\max(x,\varrho) < 2\} \cup \{y < \tfrac{1}{4}\} \cup \{y > \tfrac{3}{4}\}, \\ \mathcal{B} &= \{3\min(x,\varrho) + 5\max(x,\varrho) > 2,\ x + y > \tfrac{1}{2},\ y + z < \tfrac{3}{2}\}, \\ W &= \{0 < x < \tfrac{1}{4},\ \tfrac{1}{4} < y < \tfrac{1}{2} - x,\ y < z < \tfrac{3}{5}(1+x)\}. \end{aligned}$$

(J exchanges x and ϱ and maps y to $1 - y$.) The sets $\mathcal{C}$ and $\mathcal{B}$ are J-invariant; W is not, and JW is the opposite wing.

Lemma 12. *Up to a null set, Ω is the disjoint union of $\mathcal{C}$, $\mathcal{B}$, W and JW.*

Proof. Off the null set where one of the defining inequalities is an equality, a point outside $\mathcal{C} \cup \mathcal{B}$ has $3\min(x,\varrho) + 5\max(x,\varrho) > 2$, $\frac{1}{4} < y < \frac{3}{4}$, and either $x + y < \frac{1}{2}$ or $y + z > \frac{3}{2}$. These two alternatives are exchanged by J and are incompatible, since together they force $z - x > 1$. If $x + y < \frac{1}{2}$, then $y > \frac{1}{4}$ forces $x < \frac{1}{4}$ and $y < \frac{1}{2} - x$; if $\varrho \le x$ we would have $3\varrho + 5x \le 8x < 2$, so $\varrho > x$ and the first inequality reads $3x + 5\varrho > 2$, that is, $z < \frac{3}{5}(1+x)$. Conversely the inequalities defining W imply all of these. So the first alternative is W and the second is JW. □

Lemma 13 (Dominance)**.** *If $\kappa(x,y,z;c) > 0$ then $x \le c_k \le z$ for $k = 1,2,3$. Consequently:*

(i) *every U-profile vanishes off $\{3\min(x,\varrho) + 5\max(x,\varrho) < 2\} \subseteq \mathcal{C}$;*

(ii) *every raw V-profile h_{F_V} vanishes off $\{x+y < \frac{1}{2}\}$, so every V-profile vanishes off $\{x + y < \frac{1}{2}\} \cup \{y + z > \frac{3}{2}\}$;*

(iii) *every profile in $\mathcal{T}_2$ vanishes on $\mathcal{B}$, and on W the V-profile equals $\frac{1}{2}h_{F_V}$.*

Proof. In the notation of the proof of Lemma 11, $\kappa > 0$ requires $z \geq u \geq \max(c_1, c_2)$ and $x \leq w \leq \min(c_1, c_2)$, while $c_3 = x + (y - w) + (z - u) \geq x$ and $c_3 = z - (u - y) - (w - x) \leq z$.

(i) For the raw U-profile, $c_1 = \alpha < a_0$ and $c_2 = 1 - \beta$ with $\beta < b_0$, so $x < a_0$ and $\varrho < b_0$. If $x \leq \varrho$ then $3x + 5\varrho < 3a_0 + 5b_0 < 2$; if $\varrho < x$ then $3\varrho + 5x < 8a_0 \leq 3a_0 + 5b_0 < 2$. The set is J-invariant, so the reflected part vanishes off it too.

(ii) For V, $c_1, c_2 < \eta < \frac{1}{4}$ and $c_3 \leq z$, so $x + y = c_1 + c_2 + c_3 - z < \frac{1}{2}$. Reflection maps $\{x + y < \frac{1}{2}\}$ to $\{y + z > \frac{3}{2}\}$.

(iii) Both sets in (i) and (ii) miss $\mathcal{B}$. On W the U-profiles vanish by (i), and $y + z < \frac{1}{2} + \frac{3}{4} < \frac{3}{2}$, so the reflected half of a V-profile vanishes by (ii). □

3.5 The span

Lemma 14 (Density on $\mathcal{C}$)**.** *$L^2_J(\mathcal{C}) \subseteq \mathcal{H}(\mathcal{T}_2)$.*

Proof. Write $\mathcal{H} = \mathcal{H}(\mathcal{T}_2)$.

Step 1: point features and limits. Let $\mathcal{F}$ be the set of bounded Borel F such that $\frac{1}{2}(h_F + h_F \circ J) \in \mathcal{H}$. It is a linear space, and it is closed under uniform convergence because $\|h_F\| \leq \|F\|_\infty \|v\|$ (Lemma 6(ii)) and $\mathcal{H}$ is closed. It contains F_U and F_V for all admissible data. It also contains $f(c_1)g(c_2)P(c_3)$ whenever P is continuous on $[0, 1]$ and either $f \in C_c((0, a_0))$, $g(1 - \cdot) \in C_c((0, b_0))$ with $0 < a_0 \leq b_0$, $3a_0 + 5b_0 < 2$, or $f, g \in C_c((0, \eta))$ with $\eta < \frac{1}{4}$. Indeed, $P(c_3) = p(\frac{1}{2} - c_3)$ for a continuous p on $[-\frac{1}{2}, \frac{1}{2}]$, which is a uniform limit of polynomials; continuous compactly supported f, g are uniform limits of smooth ones; and a continuous f with compact support in $(0, \eta)$ is the uniform limit of $u(\eta - u)P_n(u)$, where the polynomials P_n approximate $f(u)/(u(\eta - u))$ uniformly on $[0, \eta]$.

Step 2: synthesis of a bump. Fix a point of $\mathcal{C}$ off the null set where $x = \varrho$, $y = \frac{1}{4}$, $y = \frac{3}{4}$ or $3\min(x, \varrho) + 5\max(x, \varrho) = 2$. Replacing it by its image under J if necessary, we may assume that it lies in $\{y < \frac{1}{4}\}$, or in $\{3x + 5\varrho < 2,\ x < \varrho\}$. Let Q be a closed box around it, inside Ω and inside the same one of these two sets, so small that its projections leave room for admissible data: in the first case $Q \subseteq \{y < \eta\}$ for some $\eta < \frac{1}{4}$; in the second, $x < a_0$ and $\varrho < b_0$ on Q for some $a_0 \leq b_0$ with $3a_0 + 5b_0 < 2$.

Let $\psi \in C_c^\infty(\operatorname{int} Q)$ and write $\hat{\psi}(m_1, m_2, m_3) = \psi(m_3, m_2, m_1)$ for its expression in descending coordinates. By (S4),

$$\sum_\lambda \psi(\lambda/N)\, s_\lambda = \sum_m \beta_N(m)\, e_{m_1} e_{m_2} e_{m_3},$$

$$\beta_N(m) = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, \hat{\psi}\Big(\frac{m + \iota - \sigma}{N}\Big),$$

where $\iota = (1,2,3)$, $\sigma = (\sigma(1), \sigma(2), \sigma(3))$, and $\hat{\psi}$ is extended by zero (for large N every argument in the support has strictly decreasing coordinates in $(0,1)$, so the substitution $m_i = \lambda'_i - i + \sigma(i)$ is a bijection on the relevant terms). Taylor expansion of $\hat{\psi}$ at m/N gives

$$\beta_N(m) = N^{-3}\Psi(m/N) + O(N^{-4}), \qquad \Psi = \frac{1}{6}\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\big((\iota - \sigma)\cdot\nabla\big)^3 \hat{\psi},$$

uniformly in m, with $\beta_N(m) = 0$ unless m/N is within $O(1/N)$ of $\operatorname{supp}\hat{\psi}$. The terms of order $0, 1, 2$ cancel because $\sum_\sigma \operatorname{sgn}(\sigma) R(\sigma) = 0$ for every polynomial R of degree at most 2 in three variables: its alternation is an alternating polynomial of degree at most 2, hence divisible by the Vandermonde polynomial of degree 3, hence zero. Taking the coefficient of s_λ and using $K_\lambda \geq 0$, $\sum_m K_\lambda(m) = d_\lambda = O(N^3)$,

$$\psi(\lambda/N) = N^{-3} c_\lambda(\Psi) + O(N^{-1}) = h_\Psi(\lambda/N) + o(1)$$

uniformly, by Lemma 6(i). As ψ and h_Ψ are continuous, $h_\Psi = \psi$ on Ω.

Step 3: approximation. Ψ is continuous with support in $\operatorname{supp}\hat{\psi}$. In the first case assign the coordinates $(m_3, m_2, m_1) \leftrightarrow (x, y, z)$ of Ψ to (c_1, c_2, c_3); in the second, to (c_1, c_3, c_2), with $c_2 = 1 - \varrho$ the reflected short factor. By the Stone–Weierstrass theorem, Ψ is a uniform limit on $[0,1]^3$ of finite sums $\sum f_k(c_1) g_k(c_2) P_k(c_3)$ of continuous functions. Multiplying the two short factors by continuous cutoffs $\chi_1(c_1)\chi_2(c_2)$ that equal 1 on the projections of $\operatorname{supp}\Psi$ and are supported where the data of Step 1 are admissible, we keep a uniform approximation of $\Psi = \Psi\chi_1\chi_2$. Since the order of the three factors in E_N is immaterial, $h_{F\circ\pi} = h_F$ for every permutation π of the coordinates, so each approximant lies in $\mathcal{F}$, and hence $\Psi \in \mathcal{F}$, that is, $\frac{1}{2}(\psi + \psi\circ J) \in \mathcal{H}$.

Step 4. Every J-symmetric $\phi \in C_c^\infty$ supported in $\mathcal{C}$ off the null set is, by a partition of unity, a finite sum of functions $\frac{1}{2}(\psi + \psi\circ J)$ as in Step 2. These are dense in $L^2_J(\mathcal{C})$, and $\mathcal{H}$ is closed. □

For a continuous g on $\mathbb{R}^2$ put $\Gamma g(x,y,z) = g(r, x+y) - g(r, x+z)$, where $r = x+y+z$. Let $\mathcal{G}$ be the closure in $L^2(W)$ of $\{\Gamma g|_W : g \in C(\mathbb{R}^2),\ g(r,u) = 0 \text{ for } u \geq \frac{1}{2} - \epsilon \text{ for some } \epsilon > 0\}$, and let $\widetilde{\mathcal{G}}$ be the set of J-symmetric functions supported in $W \cup JW$ whose restriction to W lies in $\mathcal{G}$.

Proposition 15 (The span). *$\mathcal{H}(\mathcal{T}_2) = L^2_J(\mathcal{C}) \oplus \widetilde{\mathcal{G}}$, an orthogonal sum. In particular every element of $\mathcal{H}(\mathcal{T}_2)$ vanishes on $\mathcal{B}$.*

Proof. The two summands are orthogonal since $\mathcal{C}$ and $W \cup JW$ are disjoint (Lemma 12).

$\subseteq$. Let h be a profile in $\mathcal{T}_2$. It is J-symmetric and vanishes on $\mathcal{B}$ (Lemma 13), so by Lemma 12 $h = h\mathbf{1}_{\mathcal{C}} + h\mathbf{1}_{W\cup JW}$ with $h\mathbf{1}_{\mathcal{C}} \in L^2_J(\mathcal{C})$. On W, h vanishes for U-tests, and for a V-test $h = \frac{1}{2}h_{F_V}$. For $\alpha, \beta < \eta < \frac{1}{4}$ and $(x, y, z) \in W$ we have $y > \frac{1}{4} > \max(\alpha, \beta, \alpha + \beta - y)$, so

$$H_{\alpha,\beta,p} = p\big(\tfrac{1}{2} - r + \alpha + \beta\big)\big[(\alpha + \beta - x - y)_+ - (\alpha + \beta - x - z)_+\big] \quad \text{on } W, \tag{8}$$

the cutoff $j \in [0, 1]$ being automatic where the bracket is non-zero. Integrating against $f(\alpha)g(\beta)$ shows $\frac{1}{2}h_{F_V} = \Gamma G$ on W with $G(r, u) = \frac{1}{2}\iint f(\alpha)g(\beta)p(\frac{1}{2} - r + \alpha + \beta)(\alpha + \beta - u)_+ \,\mathrm{d}\alpha\,\mathrm{d}\beta$, which is continuous and vanishes for $u \geq 2\eta$. So $h\mathbf{1}_{W\cup JW} \in \widetilde{\mathcal{G}}$, and $\mathcal{H}(\mathcal{T}_2)$, the closure of the span of such h, is contained in the closed space on the right.

$\supseteq$. By Lemma 14, $L^2_J(\mathcal{C}) \subseteq \mathcal{H}(\mathcal{T}_2)$. Let g be continuous with $g(r, u) = 0$ for $u \geq \frac{1}{2} - \epsilon$. On the compact set of values (r, u) attained on $\overline{W}$, g is a uniform limit of finite sums $\sum_k P_k(r)(b_k - u)_+$ with polynomials P_k and knots $b_k \in (0, \frac{1}{2})$: interpolate $g(r, \cdot)$ linearly at fixed knots below $\frac{1}{2} - \epsilon$ (a continuous piecewise linear function of u vanishing beyond its last knot is a combination of the $(b_k - u)_+$), and approximate the continuous coefficients by polynomials. For one term, take the point feature with $\alpha = \beta = b/2$ and $p(t) = P(\frac{1}{2} + b - t)$; by (8), its projection $h = \frac{1}{2}(H + H \circ J)$ equals $\frac{1}{2}P(r)\Gamma[(b - u)_+]$ on W. This h lies in $\mathcal{H}(\mathcal{T}_2)$: the map $(\alpha, \beta) \mapsto H_{\alpha,\beta,p} \in L^2(\Omega)$ is continuous (dominated convergence; for fixed (x, y, z) the feature is continuous in (α, β) off the lines $\alpha + \beta \in \{r, r - 1\}$), so h is the limit of the profiles of V-tests whose short polynomials f, g concentrate at $b/2$, constructed as in Step 1 of Lemma 14. Its restriction to $\mathcal{C}$ lies in $L^2_J(\mathcal{C}) \subseteq \mathcal{H}(\mathcal{T}_2)$ and it vanishes on $\mathcal{B}$, so $h\mathbf{1}_{W\cup JW} \in \mathcal{H}(\mathcal{T}_2)$. Summing, and using that the maps $(x, y, z) \mapsto (r, x + y)$ and $(r, x + z)$ are linear of rank two on the bounded set W (so uniform convergence of g implies convergence of Γg in $L^2(W)$), we obtain the J-symmetric extension of $\Gamma g|_W$ in $\mathcal{H}(\mathcal{T}_2)$. Taking closures proves the claim. □

3.6 The residual least-squares problem

By Proposition 15 and Lemma 12,

$$9!\,\|P_{\mathcal{H}(\mathcal{T}_2)}v\|^2 = 42 - \mu(\mathcal{B}) - 2 \cdot 9!\,\mathrm{dist}_{L^2(W)}(v, \mathcal{G})^2, \tag{9}$$

because $P_{\mathcal{H}}v = v\mathbf{1}_{\mathcal{C}} + P_{\widetilde{\mathcal{G}}}v$, and J-symmetric functions on $W \cup JW$ have twice their squared norm on W. We compute the two losses. All integrals below

are of polynomials over polytopes and are evaluated exactly by computer, and an independently organised computation gives the same values.

The invisible set. By J-symmetry, $\mu(\mathcal{B}) = 2\mu(\mathcal{B}\cap\{x<\varrho\})$. On $\mathcal{B}\cap\{x<\varrho\}$ the first inequality forces $\varrho > \frac14$, so $y<z<\frac34$ and $y+z<\frac32$ is automatic. In the coordinates (x,ϱ,t) with $t=y-x$ and $d=1-\varrho-x=z-x$ we have $4v^2=t^2(d-t)^2d^2$, and the region is

$$\begin{aligned}&\{0<x<\tfrac14,\ \tfrac{2-3x}{5}<\varrho<\tfrac12+x,\ \tfrac12-2x<t<d\}\\ &\quad\cup\{\tfrac14<x<\tfrac12,\ x<\varrho<1-x,\ 0<t<d\}.\end{aligned}$$

(For $x>\frac14$ the constraints $3x+5\varrho>2$ and $x+y>\frac12$ are automatic.) Integrating $\frac{9!}{2}t^2(d-t)^2d^2$ over the two pieces gives $\frac{60361491}{100000000}$ and $\frac{21}{256}$, so

$$L_0 := \mu(\mathcal{B}) = \frac{8570577}{12500000}. \tag{10}$$

The second piece is the half $\{x<\varrho\}$ of the scaled chamber $\frac14<x<y<z<\frac34$; with the factor 2 included above, its value $\frac{21}{256}$ is the mass $42/2^9$ of the whole scaled chamber, by scaling.

The fit on W. Let g be admissible. On $W_> = W\cap\{z>\frac12\}$ we have $x+z>\frac12$, so $\Gamma g = g(r,x+y)$; on $W_< = W\cap\{z<\frac12\}$ both terms may be non-zero. The values of g used on $W_>$ lie in $\{r-u>\frac12\}$, because $r-(x+y)=z$; those used on $W_<$ lie in $\{r-u<\frac12\}$, because $r-(x+y)=z<\frac12$ and $r-(x+z)=y<\frac12$. Hence the two fits are independent, and $\operatorname{dist}(v,\mathcal{G})^2=\delta_>+\delta_<$, the two squared distances on $W_>$ and $W_<$ in which g ranges over all bounded measurable functions on the respective half-planes vanishing for $u\ge\frac12$. (Such g are L^2-limits of admissible ones on the bounded set W, by truncation, smoothing and dominated convergence, the push-forwards of Lebesgue measure under the two rank-two maps being absolutely continuous.)

The large region. Write $S=x+y$ and $h=S-\frac14$. On $W_>$, $\frac12<z<\frac34$ and, at fixed (z,S), x ranges over (ℓ,h) with $\ell=\max(0,\frac{5z-3}{3})$, for $\frac14+\ell<S<\frac12$. Since $r=z+S$, the fits $g(r,S)$ are exactly the functions of (z,S), and the best fit is the mean of

$$P(x)=v(x,S-x,z)=\tfrac12(z-S+x)(z-x)(S-2x)$$

over $x\in(\ell,h)$. Thus, splitting at $z=\frac35$ where ℓ changes,

$$\begin{aligned}L_1 := 2\cdot 9!\,\delta_> &= 2\cdot 9!\int_{1/2}^{3/4}\int_{1/4+\ell}^{1/2}\Big(\int_\ell^h P^2\,\mathrm{d}x-\frac{(\int_\ell^h P\,\mathrm{d}x)^2}{h-\ell}\Big)\,\mathrm{d}S\,\mathrm{d}z\\ &= \frac{588879}{25600000}+\frac{1114119}{40960000}.\end{aligned} \tag{11}$$

The small region. Put $u = 4z - 1$, $v_0 = 4y - 1$, $w = 4x$ and $m = u + v_0 + w$. Then $W_<$ becomes $\{0 < v_0 < u < 1,\ 0 < w < 1 - v_0\}$ (the constraint $z < \frac{3}{5}(1 + x)$ is automatic), and $v = d_m(u, v_0)/128$ with

$$d_m(u, v_0) = (u - v_0)(1 + u - w)(1 + v_0 - w), \qquad w = m - u - v_0.$$

Since $r = \frac{1}{4}(m + 2)$, $x + y = \frac{1}{4}(m + 1 - u)$ and $x + z = \frac{1}{4}(m + 1 - v_0)$, the fit is $\frac{1}{128}(f_m(u) - f_m(v_0))$ with f_m an arbitrary function vanishing for $u \le c_m := \max(0, m - 1)$. At fixed $m \in (0, 2)$ the fibre is $\{(u, v_0) : 0 < v_0 < u < 1,\ u > m - 1,\ u + v_0 < m\}$. Since d_m is antisymmetric in (u, v_0), the first variation in the direction of a bounded ϕ supported in $(c_m, \min(1, m))$ is $-2 \int \phi(u) \int_0^{m-u} \big(d_m(u, v_0) - f_m(u) + f_m(v_0)\big)\, \mathrm{d}v_0\, \mathrm{d}u$, so the normal equation is

$$(m - u) f_m(u) - \int_{c_m}^{m-u} f_m(v_0)\, \mathrm{d}v_0 = b_m(u) := \int_0^{m-u} d_m(u, v_0)\, \mathrm{d}v_0 \tag{12}$$

for $c_m < u < \min(1, m)$. Here $b_m(u) = \frac{1}{2}(m - u)(4u^2 + 3u - 2mu - m)$. The equation is solved by

$$f_m(u) = 3u^2 + (1 - 2m)u + \tfrac{1}{2}c_m \quad (u > c_m), \qquad f_m(u) = 0 \quad (u \le c_m). \tag{13}$$

Indeed, for $m \le 1$ this is a direct polynomial identity; for $m > 1$ the unpinned polynomial satisfies the same identity with lower limit 0, the constant $\frac{1}{2}c_m$ makes its integral over $(0, c_m)$ vanish, and $m - u \ge c_m$ for all free u. A function satisfying the normal equation for every ϕ is the orthogonal projection, since the residual is then orthogonal to the closed subspace of fits and f_m itself is admissible. The squared norm of the fit equals its inner product with d_m, which by antisymmetry is $\int_{c_m}^{\min(1,m)} f_m b_m$. With

$$\iiint_{0<v_0<u<1,\ 0<w<1-v_0} d_m^2 = \frac{104}{945},$$

$$\int_0^2 \int_{c_m}^{\min(1,m)} f_m(u) b_m(u)\, \mathrm{d}u\, \mathrm{d}m = \frac{41}{2520} + \frac{457}{5040} = \frac{77}{720},$$

and the Jacobian 4^{-3} and scaling 128^{-2},

$$L_2 := 2 \cdot 9!\, \delta_< = \frac{9!}{2 \cdot 4^9}\Big(\frac{104}{945} - \frac{77}{720}\Big) = \frac{141}{65536}. \tag{14}$$

3.7 Proof of Theorem 1

By (9)–(14),

$$9!\,\|P_{\mathcal{H}(\mathcal{T}_2)}v\|^2 = 42 - L_0 - L_1 - L_2 = \frac{264076794201}{6400000000} = C_*.$$

By Lemma 11 the family $\mathcal{T}_2$ is faithful, so Proposition 8 gives the lower bound. Conversely, for any finite system $B_1, \dots, B_n$ from $\mathcal{T}_2$ with numerator vector $\mathbf{k}$ and Gram matrix G (the limits in Theorem 9), Lemma 11 gives $k_i = \langle v, h_i\rangle$ and $G_{ij} = \langle h_i, h_j\rangle$, so for every real $\mathbf{b}$ with $\mathbf{b}^tG\mathbf{b} > 0$,

$$\frac{9!(\mathbf{b}^t\mathbf{k})^2}{\mathbf{b}^tG\mathbf{b}} = 9!\frac{\langle v,h\rangle^2}{\|h\|^2} \le 9!\,\|P_{\mathcal{H}(\mathcal{T}_2)}v\|^2 = C_*, \qquad h = \sum b_i h_i.$$

Since $\sup_h \langle v,h\rangle^2/\|h\|^2$ over the span of all profiles is $\|P_{\mathcal{H}}v\|^2$, C_* is the supremum. □

Remark. Restricted to derivative degree at most four, the supremum is strictly smaller than C_*. Suppose finite systems of degree at most four had Rayleigh quotients $\langle v,h_n\rangle^2/\|h_n\|^2$ tending to $\|P_{\mathcal{H}}v\|^2$. Replacing h_n by $-h_n$ where necessary, we may assume $\langle v,h_n\rangle \ge 0$. Since $\langle v,h_n\rangle = \langle P_{\mathcal{H}}v, h_n\rangle$, the normalised profiles then satisfy $\|h_n/\|h_n\| - P_{\mathcal{H}}v/\|P_{\mathcal{H}}v\|\|^2 = 2 - 2\langle v,h_n\rangle/(\|h_n\|\|P_{\mathcal{H}}v\|) \to 0$, so they converge in L^2 to $P_{\mathcal{H}}v/\|P_{\mathcal{H}}v\|$. On $W_>$ every degree-four profile is, at fixed S, a polynomial of degree at most four in z (by (8) with $r = z + S$), and on a rectangle in (z,S) where $h - \ell$ is bounded below this property is closed in L^2, by projection onto the five-dimensional space on each slice. But the optimal mean in (11) has a kink at $z = \frac{3}{5}$: its z-derivative jumps by $\frac{5}{3h}\big(\frac{1}{h}\int_0^h P - P(0)\big)$, which equals $-\frac{83}{960}$ at $S = \frac{3}{8}$ and is non-zero for S in an open interval by continuity. The exact degree-four supremum is not determined here.

Remark. Exact rational evaluation of Fraccaro's 250 rows with his integer weights gives $41.0089916904\ldots$, and an expanded 397-row certificate with $\deg p \le 4$ gives $41.0298094152\ldots$. These finite certificates are not used in the proofs.

4 Short cubic tests and unequal same-side lengths

We add two kinds of tests to $\mathcal{T}_2$. Put $\varrho = 1 - z$ as before and

$$\mathcal{D}_{\mathrm{acc}} = \{3\min(x,\varrho) + 5\max(x,\varrho) < 2\} \cup \{x + y < \tfrac{1}{2}\} \cup \{y + z > \tfrac{3}{2}\},$$
$$\mathcal{A} = \{r < 1\} \cup \{r > 2\}, \qquad S_3 = \mathcal{D}_{\mathrm{acc}} \cup \mathcal{A}.$$

W-tests. For real $\phi \in C_c^\infty((0,1))$ let $D_\phi(s) = \sum_{n\ge1} d_3(n)\phi(\log n/L)n^{-s}$. The test is ΠD_ϕ, its cubic weight is $F_\phi(c) = \phi(c_1+c_2+c_3)$, and its profile is $\frac12(\phi(r)+\phi(3-r))v$. (Indeed $h_{F_\phi} = \phi(r)h_1 = \phi(r)v$, because $\kappa(x,y,z;c)$ vanishes unless $c_1+c_2+c_3 = r$.)

V'-tests. Data (a_0, b_0, f, g, p) with $a_0, b_0 > 0$, $a_0 + b_0 < \frac12$, real $f \in C_c^\infty((0,a_0))$, $g \in C_c^\infty((0,b_0))$ and a real polynomial p. The test is ΠV with $V = C_p A_f A_g$, and its cubic weight and profile are those of a V-test.

Let $\mathcal{T}_3 = \mathcal{T}_2 \cup \{W\text{-tests}\} \cup \{V'\text{-tests}\}$.

Theorem 16. *$\mathcal{T}_3$ is faithful, and $v\mathbf{1}_{S_3} \in \mathcal{H}(\mathcal{T}_3)$. Moreover*

$$\mu(S_3) = 42 - L_0 + \frac{1}{512} = \frac{8263261393}{200000000} = 41.316306965.$$

In particular $\liminf_{T\to\infty} M_6(T)/(c_3 T L^9) \ge 41.316306965$.

The V'-tests remove the losses L_1, L_2 of Section 3: they reach all of $\{x+y < \frac12\} \supseteq W$. The W-tests recover the part $\mathcal{B}\cap\mathcal{A}$ of the invisible set.

4.1 The diagonal and the stable range

For $t \in [0,\infty)^{3\times3}$ write $R(t) \in [0,\infty)^3$ for its row sums and $C(t)$ for its column sums, and $\mathrm{d}t$ for Lebesgue measure on $\mathbb{R}^9$.

Lemma 17 (Stable pairings)**.** *Let F, G be continuous on $[0,1]^3$, and suppose F vanishes on $\{|c| \ge 1-\delta\}$ for some $\delta > 0$. Then*

$$\langle h_F, h_G\rangle = \int_{[0,\infty)^{3\times3}} F(R(t))\, G(C(t))\,\mathrm{d}t, \qquad \langle v, h_F\rangle = \int_{[0,\infty)^{3\times3}} F(R(t))\,\mathrm{d}t.$$

If moreover G vanishes on $\{|c| \ge 2-\delta\}$, then $\langle h_F, h_G \circ J\rangle = 0$.

Proof. By Lemma 6(iii),

$$\langle h_F, h_G\rangle = \lim_{N\to\infty} N^{-9} \sum_{m,m'} F(m/N)G(m'/N)\langle e_m, e_{m'}\rangle_N,$$

where $e_m = e_{m_1}e_{m_2}e_{m_3}$. Only $|m| = |m'| < N$ contributes, so by (S6) the sum counts the integer tables t with $R(t) = m$, $C(t) = m'$, weighted by $F(R(t)/N)G(C(t)/N)$. This is a Riemann sum over the lattice points of the region $\{t \in [0,\infty)^9 : |t| < (1-\delta)N\}$ for a bounded integrand that is continuous there, and it converges to the integral. The case $G = 1$ gives the second formula, since $h_1 = v$. For the last claim, by (S5) every constituent of $D^3\overline{e_{m'}}$ has degree $3N - |m'| > N + \delta N > |m|$, so $\langle E_N[F], D^3\overline{E_N[G]}\rangle_N = 0$. □

Lemma 18 (Arithmetic diagonal)**.** *Let Ψ be continuous with compact support in $[0,\infty)^6$, and write $\ell(n) = (\log n_1, \log n_2, \log n_3)/L$ for $n \in \mathbb{N}^3$. Then*

$$\frac{1}{L^9} \sum_{n_1 n_2 n_3 = m_1 m_2 m_3} \frac{\Psi(\ell(n), \ell(m))}{n_1 n_2 n_3} \longrightarrow a_3 \int_{[0,\infty)^{3\times 3}} \Psi(R(t), C(t)) \, \mathrm{d}t.$$

The same holds with Ψ replaced by Ψ_L, if $\Psi_L \to \Psi$ uniformly with supports in a fixed compact set; and the left side with $|\Psi|$ in place of Ψ is $O_\Psi(1)$.

Proof. For $u, w \in \mathbb{C}^3$ with positive real parts, the Euler product gives

$$\sum_{n_1 n_2 n_3 = m_1 m_2 m_3} \frac{\prod_i n_i^{-u_i/L} \prod_j m_j^{-w_j/L}}{n_1 n_2 n_3} = \mathcal{A}\Big(\frac{u}{L}, \frac{w}{L}\Big) \prod_{i,j=1}^{3} \zeta\Big(1 + \frac{u_i + w_j}{L}\Big),$$

where $\mathcal{A}$ is an Euler product that converges absolutely and uniformly for all real parts in $(-\frac{1}{10}, \frac{1}{10})$, because the terms of order p^{-1} cancel; at the origin its local factor is $(1-X)^9 \sum_{r\ge 0} \binom{r+2}{2}^2 X^r$ with $X = 1/p$, so $\mathcal{A}(0,0) = a_3$. Since $\zeta(1+\epsilon) \sim 1/\epsilon$, the left side divided by L^9 tends to $a_3 \prod_{i,j} (u_i + w_j)^{-1}$ for fixed u, w, locally uniformly. Since

$$\prod_{i,j} (u_i + w_j)^{-1} = \int_{[0,\infty)^{3\times 3}} e^{-\sum_i u_i R_i(t) - \sum_j w_j C_j(t)} \, \mathrm{d}t,$$

the measures $\nu_L = L^{-9} \sum (n_1 n_2 n_3)^{-1} \delta_{(\log n/L, \log m/L)}$ on $[0,\infty)^6$ have Laplace transforms converging on the positive orthant to that of the push-forward of $a_3 \, \mathrm{d}t$ under $t \mapsto (R(t), C(t))$. By the continuity theorem for Laplace transforms of positive measures (after an exponential tilt, which makes them finite), ν_L converges vaguely, which is the claim. The variant with Ψ_L and the bound follow from vague convergence and a positive Laplace majorant. □

Lemma 19 (Smooth expansion)**.** *Fix $1 \le K \le 4$ and a compact set $\Gamma \subset \mathbb{C}^K$. Let $\omega \in C^\infty([0,\infty))$ equal 1 on $[0,1]$ and 0 on $[2,\infty)$. For $T \le t \le 2T$ and $\gamma \in \Gamma$,*

$$\prod_{k=1}^{K} \zeta\Big(s + \frac{\gamma_k}{L}\Big) = \sum_{m \ge 1} \frac{d_{\gamma/L}(m)}{m^s} \, \omega\Big(\frac{m}{T^8}\Big) + O_A(T^{-A}),$$

where $d_{\gamma/L}(m)$ are the coefficients of $\prod_k \zeta(s + \gamma_k/L)$, uniformly in γ, and similarly for t-derivatives of fixed order and for ∂_γ-derivatives (so for C_p in place of a shifted zeta factor).

Proof. By Mellin inversion the sum equals $\frac{1}{2\pi i}\int_{(2)}\prod_k \zeta(s+w+\gamma_k/L)\widetilde{\omega}(w)T^{8w}\,\mathrm{d}w$, where $\widetilde{\omega}(w)=\int_0^\infty \omega(x)x^{w-1}\,\mathrm{d}x$ has a simple pole at $w=0$ with residue 1 and decays faster than any power of $|\Im w|$ in vertical strips. Move the line to $\Re w=-B$. The residue at $w=0$ is the product of zeta values; the poles at $w=1-s-\gamma_k/L$ have $|\Im w|\asymp T$ and contribute $O_A(T^{-A})$. On $\Re w=-B$ the functional equation gives $|\zeta(s+w+\gamma_k/L)|\ll T^{B+1/2}$, so the integral is $O(T^{K(B+1/2)-8B})$, which is $O(T^{-A})$ for B large in terms of A, since $K\le 4$. (For $K>4$ the same argument applies with T^{2K} in place of T^8; we only use $K\in\{1,3,4\}$.) Derivatives are treated in the same way. □

Lemma 20 (Short-side diagonal rule)**.** *Let $\theta<1$ and let $B(s)=\sum_{n\le T^\theta} b(n)n^{-s}$ with $|b(n)|\ll n^\epsilon$. Let $B_t'(s)=\sum_{m\le T^K}\beta_t(m)m^{-s}$, where each $\beta_t(m)$ is smooth in $t\in[T,2T]$ with $\partial_t^j\beta_t(m)\ll_j T^{-j}m^\epsilon$ and K is fixed. Then for $\Phi\in\mathcal{W}$*

$$\int \Phi(t/T)\,B_t'(s)\,\overline{B(s)}\,\mathrm{d}t=\sum_{n\le T^\theta}\frac{\overline{b(n)}}{n}\int\Phi(t/T)\beta_t(n)\,\mathrm{d}t+O_A(T^{-A}).$$

Proof. The off-diagonal terms are $\overline{b(n)}(mn)^{-1/2}\int\Phi(t/T)\beta_t(m)(n/m)^{it}\,\mathrm{d}t$ with $m\ne n$, $n\le T^\theta$, so $|\log(m/n)|\ge\log(1+1/n)\gg T^{-\theta}$. Integrating by parts j times gives $O_j(T(T^{1-\theta})^{-j}m^\epsilon)$. Summing over the $O(T^{K+1})$ pairs and choosing j large proves the claim. □

Lemma 21 (Dual short factor)**.** *Let $g\in C_c^\infty((0,b))$ with $b<1$. Uniformly for $T\le t\le 2T$,*

$$\chi(s)\overline{A_g(s)}=\sum_{k\ge1}k^{-s}\,g\Big(\frac{\log(t/(2\pi k))}{L}\Big)+O_g(T^{-1/2}\log T).$$

Proof. $\overline{A_g(s)}=\sum_n g(\log n/L)n^{-1/2+it}$. Split $x^{-1/2}g(\log x/L)e^{it\log x}$ by a smooth dyadic partition and apply Poisson summation on each block $x\asymp X$. The frequency k contributes $\int x^{-1/2}g(\log x/L)e^{it\log x-2\pi ikx}\,\mathrm{d}x$. For $k\ge1$ with $k\asymp T/X$ the phase has a non-degenerate stationary point at $x_0=t/(2\pi k)$, where the second derivative is $\asymp T/X^2$; after rescaling $x=Xy$ the amplitude has derivatives $O_j(X^{-1/2})$, and one-dimensional stationary phase gives

$$k^{-1/2}e^{it\log(t/(2\pi k))-it-i\pi/4}g(\log x_0/L)+O_g(X^{1/2}T^{-3/2}),$$

including the partition factor at x_0; the constant phase is $-\pi/4$ because the second derivative of $t\log x - 2\pi kx$ is negative. Away from a fixed enlargement of the stationary range, including $k \le 0$, repeated integration by parts gives $O_{g,A}(X^{1/2}(T+|k|X)^{-A})$. Summing over k and the $O(L)$ blocks gives the error $O_g(T^{-1/2}\log T)$. Stirling's formula $\chi(s) = e^{-it\log(t/2\pi)+it+i\pi/4}(1+O(1/T))$ removes the phase, and the dual amplitudes have absolute sum $O(T^{1/2})$, so the Stirling error is admissible. □

4.2 The unequal-length numerator

Proposition 22. *For every V'-test, $\mathrm{N}_T(\Pi V)/\mathcal{N}_T \to \iint f(a)g(b)\,\mathcal{K}_p(a,b)\,\mathrm{d}a\,\mathrm{d}b$.*

Proof. Put $\varkappa = a_0 + b_0 < \frac{1}{2}$. By linearity and the Cauchy integral formula in a shift variable (as in Lemma 10) it suffices to treat $p(X) = e^{\zeta_0 X}$ for ζ_0 in a fixed compact set, for which $C_p(s) = e^{\zeta_0/2}\zeta(s+\zeta_0/L)$, with all estimates uniform in ζ_0. Since $\zeta(s)^3\overline{\zeta(s+\zeta_0/L)} = \chi(1-s+\bar\zeta_0/L)\zeta(s)^3\zeta(s-\bar\zeta_0/L)$ on the critical line, and $e^{\bar\zeta_0/2}\chi(1-s+\bar\zeta_0/L)/\chi(1-s) \to e^{-\bar\zeta_0/2}$ in C^j of $t/T \in [1,2]$ for each j (differentiated Stirling), it suffices to evaluate, for four scaled shifts $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_4)$ in a compact set and $\boldsymbol{\alpha} = \boldsymbol{\sigma}/L$,

$$I(\boldsymbol{\alpha}) = \int \Phi(t/T)\,\chi(1-s)\prod_{i=1}^{4}\zeta(s+\alpha_i)\,\overline{A_f(s)A_g(s)}\,\mathrm{d}t,$$

uniformly for Φ in a bounded subset of $C_c^\infty((1,2))$; the case needed is $\boldsymbol{\sigma} = (0,0,0,-\bar\zeta_0)$.

Stationary phase. Write $f_h = f(\log h/L)$, $g_k = g(\log k/L)$, $N = hk \le T^{\varkappa}$, and let $d_{\boldsymbol{\alpha}}(m)$ be the coefficients of $\prod_i \zeta(s+\alpha_i)$. Insert the expansion of Lemma 19 (with a cutoff e^{-m/T^K} instead, for K large; the difference is immaterial). The phase of $\chi(1-s)(N/m)^{it}$ is $t\log(t/2\pi) - t - \frac{\pi}{4} + t\log(N/m)$, stationary at $t_0 = 2\pi m/N$, and the amplitude of $\chi(1-s)$ is $1+O(T^{-1})$ with symbol bounds. For $m \asymp NT$ stationary phase gives $2\pi\sqrt{m/N}\,e(-m/N)\Phi(t_0/T) + O(T^{-1/2})$, where $e(x) = e^{2\pi i x}$; outside a fixed range of $m/(NT)$ integration by parts gives any power saving. Including $(mN)^{-1/2}$ and summing over $m \asymp NT$ gives an error $O(T^{\epsilon})$ per pair (h,k), so

$$I(\boldsymbol{\alpha}) = 2\pi\sum_{h,k}\frac{f_h g_k}{N}\sum_{m} d_{\boldsymbol{\alpha}}(m)e(-m/N)\Phi\Big(\frac{2\pi m}{NT}\Big) + O_\epsilon(T^{\varkappa+\epsilon}). \tag{15}$$

Characters. For $q = N/d$,

$$e(-m/N) = \sum_{d\mid(m,N)}\frac{1}{\varphi(q)}\sum_{\xi \bmod q}\tau(\bar\xi)\,\xi(-m/d).$$

After Mellin inversion of Φ, the Dirichlet series attached to the pair (d,ξ) is $\sum_n d_{\boldsymbol{\alpha}}(dn)\xi(n)(dn)^{-w}$. On $\Re w = \frac12+\delta$, $0<\delta<\frac{1}{10}$, it equals $d^{-w}\prod_i L(w+\alpha_i,\xi)$ times local corrections at the primes dividing d, whose product is $O_\epsilon(d^\epsilon)$ by the divisor bound. We use the fourth-moment bound

$$\frac{1}{\varphi(q)}\sum_{\xi \bmod q}|L(w,\xi)|^4 \ll_{\delta,\epsilon} q^\epsilon(1+|\Im w|)^{C_{\delta,\epsilon}} \qquad (\Re w = \tfrac12+\delta+O(1/L)). \tag{16}$$

To prove it, group the characters by their primitive inducing character of conductor $r \mid q$ and parity; removing the Euler factors at primes dividing q costs $O_\epsilon(q^\epsilon)$. The approximate functional equation for $L(w,\xi)^2$, ξ primitive modulo r, has two smoothed sums of length $r^{1+\epsilon}(1+|\Im w|)^C$ with coefficients $d_2(n)n^{-w}$ and $d_2(n)n^{-(1-w)}$, the latter multiplied by a factor of absolute value $\ll (r(1+|\Im w|))^{1-2\Re w}$. The large sieve for characters modulo r bounds the mean square of a block of length Y by $(r+Y)\sum|a_n|^2$; the prefactor absorbs the growth of the dual coefficients. Dyadic summation gives $O(r^{1+\epsilon}(1+|\Im w|)^C)$ for the sum over characters modulo r, and summing over $r \mid q$ gives (16); Hölder's inequality then bounds the mean of the product of four shifted L-functions. For non-principal ξ move the Mellin contour to $\Re w = \frac12+\delta$; there are no poles. Since $|\tau(\xi)|\le\sqrt{q}$, the contribution for fixed N, before the factor $1/N$ in (15), is

$$\ll_\epsilon (NT)^{1/2+\delta}\sum_{d\mid N} d^{-1/2-\delta+\epsilon}(N/d)^{1/2+\epsilon} \ll_\epsilon T^{1/2+\delta+\epsilon}N^{1+\delta+\epsilon}.$$

Summing over h,k with weight $1/N$ gives $O(T^{1/2+\delta+\epsilon+\varkappa(1+\delta+\epsilon)}) = O(T^{1-\eta})$ for some $\eta>0$, on choosing δ,ϵ small in terms of $\frac12-\varkappa$. For principal characters the same contour shift crosses only the poles at $w=1-\alpha_i$, and the new line obeys the same bound.

Residues. For the principal characters, write $w = 1+\omega$ and $D_p(w) = \prod_i(1-p^{-w-\alpha_i})^{-1}$. After summing over $d \mid N$, the local factor at $p^\nu \| N$ is

$$B_p(0,w) = D_p(w),$$
$$B_p(\nu,w) = \sum_{r\ge\nu} d_{\boldsymbol{\alpha}}(p^r)p^{-rw} - \frac{d_{\boldsymbol{\alpha}}(p^{\nu-1})p^{-(\nu-1)w}}{p-1} \quad (\nu\ge1),$$

because the principal Gauss sum modulo q is $\mu(q)$, so only $v_p(d)\in\{\nu,\nu-1\}$ survive. With Mellin variables u,v' for the profiles f,g, the Euler product of the local factors $\sum_{j,k\ge0}B_p(j+k,1+\omega)p^{(j+k)\omega-ju-kv'}$ factors as

$$G(\boldsymbol{\alpha},\omega,u,v')\prod_i\zeta(1+\omega+\alpha_i)\prod_{h\in\{u,v'\}}\frac{\prod_i\zeta(1+h+\alpha_i)}{\zeta(1+h-\omega)}, \tag{17}$$

with G absolutely and locally uniformly convergent when all real parts are at most a small δ' in absolute value: indeed $p^{\omega}B_p(1,1+\omega)/D_p(1+\omega) = \sum_i p^{-1-\alpha_i} - p^{-1+\omega} + O(p^{-2+O(\delta')})$ and the terms with $j+k \geq 2$ are $O(p^{-2+O(\delta')})$. At the origin the unfactored local series is

$$\sum_{r\geq 0}\binom{r+3}{3}\binom{r+2}{2}X^r - \frac{X}{1-X}\sum_{r\geq 0}(r+2)\binom{r+3}{3}X^r = \frac{1+4X+X^2}{(1-X)^6}$$

with $X = 1/p$, as one sees by applying $X\partial_X$ to $(1-X)^{-4}$. Removing the ten net polar factors leaves $(1-X)^4(1+4X+X^2)$, so $G(0,0,0,0) = a_3$.

First let the σ_i be pairwise distinct. At $\omega = -\alpha_j$ the two denominators in (17) cancel the j-th numerator factors. With

$$F_j(x) = \mathcal{L}^{-1}_{u\to x}\prod_{i\neq j}(u+\sigma_i)^{-1}, \qquad S(\boldsymbol{\sigma};x,y) = \sum_{j=1}^{4}\frac{e^{-\sigma_j}F_j(x)F_j(y)}{\prod_{i\neq j}(\sigma_i-\sigma_j)},$$

the residue sum divided by a_3TL^9 tends to $\int\Phi\cdot\iint f(x)g(y)S(\boldsymbol{\sigma};x,y)\,\mathrm{d}x\,\mathrm{d}y$: the three zeta factors at the shift differences give $L^3/\prod_{i\neq j}(\sigma_i-\sigma_j)$, the remaining six give L^6 times the inverse Laplace transforms (dominated on contours $\Re u = \Re v' = c/L$ by the rapid decay of the transforms of f,g), $(T/2\pi)^{-\alpha_j}\to e^{-\sigma_j}$ and $\widetilde{\Phi}(1-\alpha_j)\to\widetilde{\Phi}(1) = \int\Phi$. To remove the separation, write $Q(u) = \prod_i(u+\sigma_i)$ and $F(x,v) = \mathcal{L}^{-1}_{u\to x}((u-v)/Q(u))$; then $F(x,-\sigma_j) = F_j(x)$ and the residue theorem gives

$$S(\boldsymbol{\sigma};x,y) = \frac{1}{2\pi i}\oint\frac{e^{v}F(x,v)F(y,v)}{Q(v)}\,\mathrm{d}v$$

over a circle enclosing the roots of Q. This is holomorphic in $\boldsymbol{\sigma}$, and the convergence extends from separated shifts to all shifts in a compact set by the maximum principle on polydiscs.

Identification. For $\boldsymbol{\sigma} = (0,0,0,-z)$, $z = \bar{\zeta}_0$, we have $Q(u) = u^3(u-z)$, and an exact computation gives, for $0<x\leq y$ with $x+y<1$,

$$e^{-z/2}S(\boldsymbol{\sigma};x,y) = \mathcal{K}_{\exp(z\,\cdot)}(x,y),$$

both sides being computed in closed form; the right side is evaluated by the Hermite–Genocchi formula $\mathbb{E}\exp(z\sum_i c_iX_i) = 6z^{-3}\,[c_0,c_1,c_2,c_3]\exp(z\,\cdot)$ for the third divided difference. Both sides are entire in z and symmetric in (x,y). Collecting the factor $e^{-\bar{\zeta}_0/2}$ from the reduction, taking real parts and differentiating in ζ_0 at 0 gives the claim for every polynomial p. $\square$

Remark. This recovers, with a different organisation, the four-residue calculation of Durkan and Page [8, §5.2] for same-side amplifiers, with the product of the two lengths, rather than their maximum, controlling the error. Dirichlet GRH would not improve the exponent of N in the non-principal estimate, so the argument is the same with or without it.

4.3 Proof of Theorem 16

Faithfulness. Pairs within $\mathcal{T}_2$ are covered by Lemma 11. We use $\langle \Pi H, \Pi K\rangle = \frac{1}{2}\operatorname{Re}\langle H, K\rangle + \frac{1}{2}\operatorname{Re}\langle H, \mathcal{J}K\rangle$ for real-weighted pairings, where $\mathcal{J}K = \chi^3\overline{K}$, which holds because $\langle \mathcal{J}H, \mathcal{J}K\rangle = \overline{\langle H, K\rangle}$ and $\langle \mathcal{J}H, K\rangle = \overline{\langle H, \mathcal{J}K\rangle}$; on the Schur side the corresponding identity is $\langle \frac{1}{2}(h + h\circ J), \frac{1}{2}(h' + h'\circ J)\rangle = \frac{1}{2}\langle h, h'\rangle + \frac{1}{2}\langle h, h'\circ J\rangle$.

(a) *Pairings with a W-test.* Let $\theta < 1$ with $\operatorname{supp}\phi \subset (0,\theta)$, so D_ϕ has length $\le T^\theta$. Every other test K in $\mathcal{T}_3$, and ζ^3, is, after Lemma 19 for its zeta factors and Lemma 21 for a reflected short factor $\chi\overline{A_g}$, of the form B'_t in Lemma 20, up to an error whose weighted L^1-norm against $|D_\phi|$ is $o(TL^9)$. For U this uses $A_f \ll T^{a_0/2}$, $C_p \ll_\epsilon T^{1/4+\epsilon}$ (convexity and Cauchy's formula), $\int \Phi|D_\phi| \ll TL^{9/2}$ and $-\frac{1}{2} + \frac{a_0}{2} + \frac{1}{4} < 0$, since $a_0 < \frac{1}{4}$. The diagonal of Lemma 20 imposes $n = m_1m_2m_3$ with $n \le T^\theta$, and on it the dual profile $g(\log(t/(2\pi k))/L)$ may be replaced by $g(1 - \log k/L)$ at a cost $O(1/L)$ per term, hence $O(TL^8)$ in total by Lemma 18. Then Lemma 18 evaluates the direct pairings $\langle D_\phi, K\rangle$, $\langle \zeta^3, D_\phi\rangle$ as $a_3TL^9\int\Phi$ times $\int \phi(|R(t)|)G(C(t))\,\mathrm{d}t$, where G is the cubic weight of K (respectively $\int\phi(|R(t)|)\,\mathrm{d}t$), and by Lemma 17 these are $\langle h_{F_\phi}, h_G\rangle$ and $\langle v, h_{F_\phi}\rangle$. The reflected pairings: $\langle D_\phi, \mathcal{J}D_\psi\rangle$ has no stationary point, since the product of the two indices is $< T^2$ while the phase of χ^3 requires order T^3, so it is $O(T^{-A})$; $\langle D_\phi, \mathcal{J}V\rangle$ is negligible because after $\chi\overline{C_p} = C_{p^-} + O(\cdot)$ (the centred functional equation, with squared weighted error $O(TL^7)$ as in [9, §2.3]) the remaining phase has derivative $-2\log(t/2\pi) - \log m + \log(nhk) + O(1/T)$ with $nhk \le T^{\theta+1/2}$; and $\mathcal{J}U = \chi^2\overline{C_p}\,\overline{A_f}A_g$ is, by the same two replacements, a U-type product with (f,g,p) replaced by (g,f,p^-), to which the direct argument applies. On the Schur side Lemma 17 gives $\langle h_{F_\phi}, h_G\circ J\rangle = 0$ whenever G vanishes on $|c| \ge 2-\delta$, which covers F_ψ and F_V, and for U it gives the profile of the reflected data, in agreement.

(b) *V'-tests.* The numerator is Proposition 22. For Gram entries among U, V and V'-tests, Fraccaro's evaluation [9, §2.3] uses only that the four short exponents of the two tests sum to less than 1, which holds for every pair here (for U with V': $a_0 + b_0 + \varkappa < \frac{1}{2} + \frac{1}{2}$); the limits are therefore those of

Theorem 9, and the proof of Lemma 11, which used only this condition, identifies them with Schur inner products.

The span. The proof of Lemma 14 applies verbatim on boxes in $\{x+y<\frac{1}{2}\}$, assigning the short factors of a V'-test to the coordinates x, y (with a_0, b_0 slightly above their ranges and $a_0+b_0<\frac{1}{2}$) and the derivative to z; with Lemma 14 and reflection this gives $L^2_J(\mathcal{D}_{\mathrm{acc}})\subseteq\mathcal{H}(\mathcal{T}_3)$. Let $0\le\phi_n\le 1$ in $C_c^\infty((0,1))$ increase to $\mathbf{1}_{(0,1)}$. The profiles $\frac{1}{2}(\phi_n(r)+\phi_n(3-r))v$ of the W-tests converge in L^2 to $\frac{1}{2}v\mathbf{1}_{\mathcal{A}}$, so $v\mathbf{1}_{\mathcal{A}}\in\mathcal{H}(\mathcal{T}_3)$. Since $v\mathbf{1}_{\mathcal{A}\cap\mathcal{D}_{\mathrm{acc}}}\in L^2_J(\mathcal{D}_{\mathrm{acc}})$, we get $v\mathbf{1}_{S_3}=v\mathbf{1}_{\mathcal{D}_{\mathrm{acc}}}+v\mathbf{1}_{\mathcal{A}}-v\mathbf{1}_{\mathcal{A}\cap\mathcal{D}_{\mathrm{acc}}}\in\mathcal{H}(\mathcal{T}_3)$.

The measure. Up to null sets $\Omega\setminus\mathcal{D}_{\mathrm{acc}}=\mathcal{B}$, by the definitions, so $\Omega\setminus S_3=\mathcal{B}\setminus\mathcal{A}$ and $\mu(S_3)=42-\mu(\mathcal{B})+\mu(\mathcal{B}\cap\mathcal{A})$. On $\{r<1\}$, $\mathcal{B}$ is $E=\{r<1,\ x+y>\frac{1}{2}\}$: there $z<\frac{1}{2}$, so $\varrho>\frac{1}{2}>x$, $3x+5\varrho>2$ and $y+z<\frac{3}{2}$ hold automatically. By J-symmetry $\mu(\mathcal{B}\cap\mathcal{A})=2\mu(E)$. The substitution $b_1=y+z-\frac{1}{2}$, $b_2=x+z-\frac{1}{2}$, $b_3=x+y-\frac{1}{2}$ maps E onto $\{b_1>b_2>b_3>0,\ \sum b_i<\frac{1}{2}\}$, preserves the three pairwise differences up to order, and has Jacobian 2. Hence $\mu(E)=\frac{1}{2}\cdot 2^{-9}\mu(\{r<1\})$, and $\mu(\{r<1\})=9!\int_{|t|<1}\mathrm{d}t=1$ by Lemma 17 with $F=\mathbf{1}_{\{|c|<1\}}$ (approximated by continuous functions). So $2\mu(E)=\frac{1}{512}$, which is also checked by direct integration. Together with (10) this gives $\mu(S_3)$, and Proposition 8 gives the lower bound. □

Remark. One can also adjoin, on top of $\mathcal{T}_3$, three-factor tests just past total degree one, evaluated with Heath-Brown's theorem on d_3 in progressions [15]. Their gain lies, inside the region N_0 of Theorem 23, in a strip $\{40r+21k<51\}$ that is contained in the balanced region $\mathcal{R}$ of Theorem 52. They therefore do not affect Theorem 2, and we omit them.

5 Marked tests and the spectral estimate

In this section we enlarge the support from S_3 to $S_3\cup\mathcal{Z}$, where

$$\mathcal{Z}=\{(x,y,z)\in\Omega:\ z-x>\tfrac{1}{2}\}$$

is the *spread region*. It is J-invariant, since J preserves $z-x$. Write $\mathcal{D}_+=\{x+y<z\}$ for the dominant shapes. Every point of $\mathcal{Z}$ lies in $\mathcal{D}_+\cup J\mathcal{D}_+$: as $J\mathcal{D}_+=\{y>1-(z-x)\}$ and $1-(z-x)<\frac{1}{2}<z-x$, a point with $y\ge z-x$ has $y>1-(z-x)$.

We use the coordinates (r,k,x), $r=x+y+z$, $k=x+y$, on Ω; the map $(x,y,z)\mapsto(r,k,x)$ has Jacobian ±1. Let

$$\mathcal{Y}=\Big\{(r,k,x):\ 1<r<2,\ \max(r-1,\tfrac{1}{2})<k<\tfrac{r}{2},\ 0<x<\tfrac{k}{2},\ r-k-x>\tfrac{1}{2}\Big\},$$

an open set. For a function ψ on $\mathcal{Y}$ put $\psi^\flat(x,y,z)=\psi(x+y+z,x+y,x)$, extended by 0. The conditions defining $\mathcal{Y}$ say exactly that $(x,y,z)=(x,k-x,r-k)\in\Omega\cap\mathcal{D}_+\cap\mathcal{Z}$ with $1<r<2$ and $k>\frac12$: indeed $x<k/2$ is $x<y$, $k<r/2$ is $x+y<z$ (and implies $y<z$), $k>r-1$ is $z<1$, and $r-k-x>\frac12$ is $z-x>\frac12$.

Theorem 23. *For real $\psi\in C_c^\infty(\mathcal{Y})$ let W_ψ be the marked test of Definition 24 below. Let*

$$\mathcal{T}_{\mathcal{M}}=\big\{(W_\psi,\psi^\flat),\ (\mathcal{J}W_\psi,\psi^\flat\circ J):\ \psi\in C_c^\infty(\mathcal{Y})\ \textit{real}\big\},\qquad \mathcal{T}_4=\mathcal{T}_3^\infty\cup\mathcal{T}_{\mathcal{M}},$$

where $\mathcal{J}B=\chi(s)^3\overline{B}$ and $\mathcal{T}_3^\infty$ is the set of U-, V'- and W-tests in $\mathcal{T}_3$ (all of which have smooth data). Then $\mathcal{H}(\mathcal{T}_3^\infty)=\mathcal{H}(\mathcal{T}_3)$, $\mathcal{T}_4$ is faithful, $v\mathbf{1}_{S_3\cup\mathcal{Z}}\in\mathcal{H}(\mathcal{T}_4)$, and

$$\mu(S_3\cup\mathcal{Z})=42-2\mu(N_0)=42-\frac{209}{512}=41.591796875,$$

where $N_0=\{(x,y,z)\in\Omega:\ 1<r<\frac32,\ k>\frac12,\ z-x<\frac12\}$.

The proof occupies the rest of the section; it is completed in Section 5.7. Its analytic core is a mixed pairing (Proposition 38) between a marked test and an arbitrary smooth cubic source, which rests on the signed estimate Theorem 39 for the ternary divisor function on rational arcs. That estimate is proved with the Kuznetsov formula at an Atkin–Lehner cusp of $\Gamma_0(M)$, where the numerator phase disappears and the shifted divisor coefficients factor through three standard L-functions; the levels M are then averaged with the large sieves for exceptional eigenvalues of Deshouillers–Iwaniec and Watt.

Throughout this section, Φ denotes a smooth function supported in a fixed compact subset of $(1,2)$, possibly complex-valued, and statements are uniform for Φ in bounded subsets of C_c^∞ (with fixed support). We write

$$\langle B,B'\rangle_\Phi=\int_{\mathbb{R}}\Phi(t/T)\,B(\tfrac12+it)\,\overline{B'(\tfrac12+it)}\,\mathrm{d}t,\qquad \|B\|_\Phi^2=\langle B,B\rangle_{|\Phi|}.$$

For $\Phi\in\mathcal{W}$ this is $\mathrm{G}_T(B,B')$. All implied constants may depend on ψ, the support of Φ, and fixed smooth weights; $\epsilon>0$ is arbitrarily small and may change between occurrences.

5.1 Marked witnesses

Definition 24. Let $\psi\in C_c^\infty(\mathcal{Y})$ be real and put $m_\psi(r,k,u)=-\partial_3\psi(r,k,u)$. For $q\ge1$ and $r\in\mathbb{R}$ let

$$\beta_\psi(q;r)=\sum_{lde=q}\frac{\mu(l)}{l}\,m_\psi\Big(r,\frac{\log q}{L},\frac{\log d}{L}\Big),$$

and for $n \geq 1$

$$M_\psi(n) = L^2 \sum_{q \geq 1} \frac{c_q(n)}{q}\, \beta_\psi\Big(q; \frac{\log n}{L}\Big), \qquad W_\psi(s) = \sum_{n \geq 1} M_\psi(n) n^{-s},$$

where $c_q(n) = \sum^*_{a \bmod q} e(an/q)$ is Ramanujan's sum.

The sums are finite: $m_\psi(r, \cdot, \cdot) = 0$ unless r lies in a compact subset $I_\psi \subset (1, 2)$, so $n \leq T^2$, and then $q \leq T^{r/2-\delta}$ for some $\delta = \delta_\psi > 0$. Since $|\beta_\psi(q; r)| \leq \|m_\psi\|_\infty \tau_3(q)$ and $|c_q(n)| \leq (q, n)$, we have $M_\psi(n) \ll_\epsilon n^\epsilon$. More precisely, on the support of $\beta_\psi(q; \log n/L)$,

$$T^{r-1+\delta} \leq q \leq T^{r/2-\delta}, \qquad r = \frac{\log n}{L}, \tag{18}$$

because $k = \log q/L$ ranges over a compact subset of $(r - 1, r/2)$. With $X = T^r$ and $H = X/T$ this says $HX^\delta \leq q \leq X^{1/2-\delta}$.

The *cubic weight* of W_ψ is

$$c_\psi(a_1, a_2, a_3) = (\partial_u \partial_v f_\psi)(a_1 + a_2 + a_3, a_1, a_2), \quad f_\psi(r, u, v) = m_\psi(r, u + v, u), \tag{19}$$

the derivatives being taken with r fixed. Since $m_\psi(r, k, u)$ vanishes unless $0 < u < k/2$ and $r - 1 < k < r/2 < 1$, the function c_ψ is smooth with compact support in $(0, 1)^3$: on its support $a_1 = u$, $a_2 = k - u$ and $a_3 = r - k$ lie in $(0, 1)$.

Lemma 25 (Marked profile). $h_{c_\psi} = \psi^\flat$.

Proof. Fix $(x, y, z) \in \Omega$ and put $r = x + y + z$, $F(a_1, a_2) = f_\psi(r, a_1, a_2) = m(r, a_1 + a_2, a_1)$, writing $m = m_\psi$. Since c_ψ has compact support in $(0, 1)^3$, (2) reads $h_{c_\psi}(x, y, z) = \iint \partial_1 \partial_2 F(a_1, a_2)\, \kappa(x, y, z; a)\, \mathrm{d}a_1\, \mathrm{d}a_2$ with $a_3 = r - a_1 - a_2$. By definition κ is the length of the set of $u \in [y, z]$ with $w := a_1 + a_2 - u \in [x, y]$ and $\max(a_1, a_2) \leq u$; the last condition is equivalent to $w \leq a_1 \leq u$ given $u + w = a_1 + a_2$. Changing variables $(a_1, a_2, u) \mapsto (a_1, w, u)$, which has Jacobian 1,

$$h_{c_\psi}(x, y, z) = \int_y^z \int_x^y \Lambda(u, w)\, \mathrm{d}w\, \mathrm{d}u, \quad \Lambda(u, w) = \int_w^u (\partial_1 \partial_2 F)(a, u + w - a)\, \mathrm{d}a.$$

Now $\partial_1 \partial_2 F(a_1, a_2) = (m_{kk} + m_{kx})(r, a_1 + a_2, a_1)$, where subscripts denote partial derivatives in the second and third slots, so with $k' = u + w$,

$$\Lambda(u, w) = \partial_k^2 \Big[\int_w^u m(r, k, a)\, \mathrm{d}a \Big]_{k=k'} + m_k(r, k', u) - m_k(r, k', w).$$

For $u > w$ we have $u > k'/2$, and $m(r,k,a) = 0$ for $a \geq k/2$; hence for k near k' the bracket equals $\int_w^{k/2} m(r,k,a)\,\mathrm{d}a = \psi(r,k,w)$, and $m_k(r,k',u) = 0$. Thus $\Lambda(u,w) = (\psi_{kk} + \psi_{kx})(r, u+w, w) = \partial_w \partial_u G(u,w)$ with $G(u,w) = \psi(r, u+w, w)$, and

$$h_{c_\psi}(x,y,z) = G(z,y) - G(z,x) - G(y,y) + G(y,x).$$

Here $G(z,y) = \psi(r, y+z, y)$ and $G(z,x) = \psi(r, x+z, x)$ vanish because $y + z > r/2$ and $x + z > r/2$, while $G(y,y) = \psi(r, 2y, y)$ vanishes because $\psi(r,k,\cdot)$ vanishes near $k/2$. The remaining term is $G(y,x) = \psi(r, x+y, x) = \psi^\flat(x,y,z)$. □

The identity has also been checked by numerical quadrature.

We shall also need the effect of reflection on profiles. For a bounded Borel F on $[0,1]^3$ put $F^J(c) = F(1-c_1, 1-c_2, 1-c_3)$.

Lemma 26. *$h_{F^J} = h_F \circ J$ for continuous F.*

Proof. By (S5), $D\overline{e_i} = e_{N-i}$, so $D^3\overline{E_N[F]} = E_N[F^J]$ up to the factor $F^J(m/N)$ versus $F(1-m/N)$, which agree. Hence $c_\lambda(F^J) = \langle D^3\overline{E_N[F]}, s_\lambda\rangle_N = \overline{\langle E_N[F], D^3\overline{s_\lambda}\rangle_N} = c_{\lambda^*}(F)$, using that $D^{3-\cdot}$ is antiunitary and $D^3\overline{s_\lambda} = s_{\lambda^*}$. Since $\lambda^*/N = J(\lambda/N)$, Lemma 6(i) gives the claim. □

5.2 Ramanujan polynomials

The norms and pairings of marked tests with one another are explicit; no estimate for d_3 is involved. We first record the arithmetic of the denominators.

Lemma 27 (Local factors)**.** *For complex α, β with positive real parts, $q \geq 1$ and $n \geq 1$,*

$$\sum_{d,e\geq 1:\ de\mid n} d^{-\alpha} e^{-\beta} = \zeta(1+\alpha)\zeta(1+\beta) \sum_{q\geq 1} \frac{c_q(n)}{q}\, \mathcal{D}_q(\alpha,\beta), \tag{20}$$

where $\mathcal{D}_q(\alpha,\beta) = \sum_{lde=q} \mu(l) l^{-1-\alpha-\beta} d^{-\alpha} e^{-\beta}$, the right side converging absolutely. Moreover

$$\sum_{j\geq 0} \mathcal{D}_{p^j}(\alpha,\beta) Y^j = \frac{1 - p^{-1-\alpha-\beta} Y}{(1-p^{-\alpha}Y)(1-p^{-\beta}Y)}, \tag{21}$$

and $q^{-1}\mathcal{D}_q(\alpha,\beta)\zeta(1+\alpha)\zeta(1+\beta) = \sum_{d,e\geq 1:\, q\mid de} d^{-1-\alpha} e^{-1-\beta}$.

Proof. (21) is immediate from the definition, since $\mu(l) \neq 0$ only for $l \in \{1, p\}$ at p. For the last identity, both sides are multiplicative in q after dividing by $\zeta(1+\alpha)\zeta(1+\beta)$: the right side divided by it is $\prod_p \Pr(p^{v_p(q)} \mid UV)$, where U, V are independent random positive integers with $\Pr(U = d) = d^{-1-\alpha}/\zeta(1+\alpha)$ (as formal weights), whose valuations are independent and geometric with parameters $P = p^{-1-\alpha}$ and $R = p^{-1-\beta}$. Then $\sum_{j\geq 0}(pY)^j \Pr(v_p(UV) \geq j)$ equals $[1 - pY(1-P)(1-R)/((1-pPY)(1-pRY))]/(1-pY)$, which is (21) (a rational-function identity, also checked by computer). Finally, $\mathbf{1}_{b|n} = b^{-1}\sum_{q|b} c_q(n)$ gives

$$\sum_{de|n} \frac{1}{d^\alpha e^\beta} = \sum_{d,e} \frac{1}{d^{1+\alpha}e^{1+\beta}} \sum_{q|de} c_q(n) = \sum_q c_q(n) \sum_{q|de} \frac{1}{d^{1+\alpha}e^{1+\beta}},$$

the interchange being justified by the bound $|c_q(n)| \leq (q, n)$ and the convergence of $\sum_b \tau(b)^2 b^{-1-\min(\Re\alpha, \Re\beta)}$. □

Lemma 28 (Four-edge limit). *Let $m_0 \in \{\varphi, \mathrm{id}\}$, and let w, $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{b} = (b_1, b_2)$ be complex with real parts in $(-\frac{1}{10}, \frac{1}{10})$. If moreover $\Re(w + a_i + b_j) > 0$ for all i, j, the left side of (22) converges absolutely and*

$$\sum_{q\geq 1} \frac{m_0(q)}{q^{2+w}} \mathcal{D}_q(a_1, a_2)\, \mathcal{D}'_q(b_1, b_2) = \mathcal{G}_{m_0}(w, \boldsymbol{a}, \boldsymbol{b}) \prod_{i,j=1}^{2} \zeta(1 + w + a_i + b_j), \quad (22)$$

where $\mathcal{D}'_q(b_1, b_2) = \sum_{lde=q} \mu(l) l^{-1} d^{-b_1} e^{-b_2}$ and $\mathcal{G}_{m_0}$ is given by an Euler product converging absolutely and uniformly when all real parts are at most $\frac{1}{10}$ in absolute value, uniformly in the imaginary parts, and $\mathcal{G}_\varphi(0, 0, 0) = a_3$. The right side of (22) continues the left side meromorphically to the whole region where all real parts lie in $(-\frac{1}{10}, \frac{1}{10})$; below, (22) is only used where the left side converges absolutely. The same holds with $\mathcal{D}'$ in place of $\mathcal{D}$ in the first factor, with the same value at the origin.

Consequently, let Ψ be smooth with compact support in $(0,\infty)^3$, let $a_1, a_2 \in i\mathbb{R}$ be fixed, and write, for $t = (t_{ij}) \in [0,\infty)^{2\times 2}$, $R_i = t_{i1} + t_{i2}$, $C_j = t_{1j} + t_{2j}$ and $|t| = \sum t_{ij}$. Then, as $L \to \infty$,

$$\frac{1}{L^4}\sum_{q\ge 1}\frac{\varphi(q)}{q^2}\,\mathcal{D}_q\Big(\frac{a_1}{L},\frac{a_2}{L}\Big)\sum_{lde=q}\frac{\mu(l)}{l}\,\Psi\Big(\frac{\log q}{L},\frac{\log d}{L}\Big)$$
$$\longrightarrow a_3\int_{[0,\infty)^{2\times 2}} e^{-a_1R_1-a_2R_2}\,\Psi(|t|,C_1)\,\mathrm{d}t, \quad (23)$$

$$\frac{1}{L^4}\sum_{q\ge 1}\frac{\varphi(q)}{q^2}\sum_{l'd'e'=q}\frac{\mu(l')}{l'}\sum_{lde=q}\frac{\mu(l)}{l}\,\Psi\Big(\frac{\log q}{L},\frac{\log d'}{L},\frac{\log d}{L}\Big)$$
$$\longrightarrow a_3\int_{[0,\infty)^{2\times 2}}\Psi(|t|,R_1,C_1)\,\mathrm{d}t, \quad (24)$$

where in (23) Ψ is a function of two variables. The left sides with absolute values inside are $O_\Psi(1)$.

Proof. At a prime p the left side of (22) has degree-one coefficient $m_0(p)p^{-2-w}(p^{-a_1}+p^{-a_2}-p^{-1-a_1-a_2})\times(p^{-b_1}+p^{-b_2}-p^{-1}) = \sum_{i,j}p^{-1-w-a_i-b_j}+O(p^{-2+4/10})$, which matches the degree-one coefficient of $\prod_{i,j}\zeta(1+w+a_i+b_j)$; by (21) the higher coefficients are $O((j+1)^2p^{-j(1-4/10)})$ in modulus, uniformly in the imaginary parts. Hence the local factors of $\mathcal{G}_{m_0}$ are $1+O(p^{-6/5})$ uniformly, the second-degree terms giving the main error; this is summable over p, which proves the first claim. At the origin $\mathcal{D}_{p^j}(0,0) = \mathcal{D}'_{p^j}(0,0)$ is 1 for $j=0$ and $j+1-j/p$ for $j\ge 1$ (from (21)), and with $P = 1/p$ the local factor of the left side of (22) for $m_0=\varphi$ is $1+(1-P)\sum_{j\ge1}P^j(1+j(1-P))^2$; multiplying by $(1-P)^4$ gives $(1-P)^9\sum_{j\ge0}\binom{j+2}{2}^2P^j$ (an identity of power series, both sides being $(1-P)^4(1+4P+P^2)$; also checked by computer). So $\mathcal{G}_\varphi(0,0,0)=a_3$.

For (24) fix $w_0>0$ and write

$$\Psi(k,\varrho,\upsilon)e^{w_0k} = (2\pi)^{-3}\int\widehat{\Psi}_{w_0}(\xi)e^{i(\xi_1k+\xi_2\varrho+\xi_3\upsilon)}\,\mathrm{d}\xi$$

with $\widehat{\Psi}_{w_0}$ a Schwartz function. Then the left side of (24) equals

$$(2\pi)^{-3}\int\widehat{\Psi}_{w_0}(\xi)\,L^{-4}\,\Sigma\Big(\frac{w_0-i\xi_1}{L};\Big(\frac{-i\xi_2}{L},0\Big),\Big(\frac{-i\xi_3}{L},0\Big)\Big)\,\mathrm{d}\xi,$$

where $\Sigma(w;\boldsymbol{a},\boldsymbol{b})$ is the left side of (22) with $m_0=\varphi$ and $\mathcal{D}'$ in both places; the interchange is justified by absolute convergence for $\Re w = w_0/L > 0$. By (22)

the integrand is bounded by $|\widehat{\Psi}_{w_0}(\xi)| \sup |\mathcal{G}_\varphi| \, (L^{-1}\zeta(1+w_0/L))^4 \ll |\widehat{\Psi}_{w_0}(\xi)|$, since $|\zeta(1+\sigma+i\tau)| \le \zeta(1+\sigma)$ and all shifts other than w are purely imaginary. For fixed ξ it converges, as $L \to \infty$, to $\widehat{\Psi}_{w_0}(\xi)\, a_3 \prod_{i,j}(w_0 - i\xi_1 + \tilde{a}_i + \tilde{b}_j)^{-1}$ with $\tilde{\boldsymbol{a}} = (-i\xi_2, 0)$ and $\tilde{\boldsymbol{b}} = (-i\xi_3, 0)$, by continuity of $\mathcal{G}_\varphi$ at the origin and $L^{-1}\zeta(1 + z/L) \to 1/z$. Since

$$\prod_{i,j}(w + \tilde{a}_i + \tilde{b}_j)^{-1} = \int_{[0,\infty)^{2\times 2}} e^{-w|t| - \sum_i \tilde{a}_i R_i - \sum_j \tilde{b}_j C_j} \, \mathrm{d}t \qquad (\Re w > 0),$$

dominated convergence and Fourier inversion give (24). The proof of (23) is the same, with $\mathcal{D}_q(a_1/L, a_2/L)$ in the first factor, $\tilde{\boldsymbol{a}} = (a_1, a_2)$, and only the two Fourier variables ξ_1, ξ_3. The bounds with absolute values follow from the same Euler products with all shifts real and positive. □

Lemma 29 (Ramanujan polynomials)**.** *Let $\delta > 0$ and let $\beta(q;r)$, $\beta'(q;r)$ be functions, smooth in r, with $\partial_r^j \beta(q;r) \ll_j \tau_3(q)^C$ uniformly, vanishing unless r lies in a fixed compact $I \subset (1,2)$ and*

$$T^{r-1+\delta} \le q \le T^{r/2-\delta}. \tag{25}$$

Put $M(n) = L^2 \sum_q q^{-1} c_q(n) \beta(q; \log n/L)$, and similarly M'. Then

$$\langle P[M], P[M'] \rangle_\Phi = TL^5 \Big(\int \Phi \Big) \sum_{q \ge 1} \frac{\varphi(q)}{q^2} \int \beta(q;r) \overline{\beta'(q;r)} \, \mathrm{d}r + O(T^{1-\delta/2} L^C),$$

where $P[M](s) = \sum_n M(n) n^{-s}$. In particular, taking for Φ a fixed smooth majorant $\Phi^+ \ge |\Phi|$ with support in a fixed compact subset of $(1,2)$,

$$\|P[M]\|_\Phi^2 \le \langle P[M], P[M] \rangle_{\Phi^+} \ll TL^5 \sum_q \frac{\varphi(q)}{q^2} \sup_r |\beta(q;r)|^2 + O(T^{1-\delta/2} L^C).$$

One-sided bound. *Suppose instead that β vanishes unless $r \in I$ and $q \le T^{r/2-\delta}$, with no lower bound on q. Then there is $c = c(I, \delta) > 0$ such that*

$$\|P[M]\|_\Phi^2 \le \langle P[M], P[M] \rangle_{\Phi^+} \ll TL^5 \sum_q \frac{1}{q} \sup_r |\beta(q;r)|^2 + O(T^{1-c}),$$

where the implied constants depend only on I, δ, Φ^+ and the constants in the bounds for $\partial_r^j \beta$.

Proof. Write $\langle P[M], P[M']\rangle_\Phi = T\sum_{n,m} M(n)\overline{M'(m)}(nm)^{-1/2}\widehat{\Phi}\big(\frac{T}{2\pi}\log\frac{n}{m}\big)$ with $\widehat{\Phi}(\xi)=\int\Phi(x)e(-x\xi)\,\mathrm{d}x$. Insert a smooth partition $\sum_j V(m/2^j)=1$, $V\in C_c^\infty([\frac12,2])$, and consider one block $m\asymp X=2^j$, with $X=T^{r_0}$, $r_0\in I+[-\frac1L,\frac1L]$; put $H=X/T$. As $\widehat\Phi$ is Schwartz and $M,M'\ll n^\epsilon$, pairs with $|n-m|>HX^\epsilon$ contribute $O(T^{-A})$.

Since $\overline{M'(m)}=L^2\sum_q q^{-1}c_q(m)\overline{\beta'(q;\log m/L)}$, we need, for fixed n and q, the sum $\sum_m c_q(m)G_n(m)$ with

$$G_n(y)=\overline{\beta'(q;\tfrac{\log y}{L})}\,V\Big(\frac{y}{X}\Big)(ny)^{-1/2}\,\widehat\Phi\Big(\frac{T}{2\pi}\log\frac{n}{y}\Big).$$

As $c_q(m)=\sum_{a\bmod q}^{*}e(am/q)$ and $\sum_m e(am/q)G(m)=\sum_{\nu\in\mathbb{Z}}\widehat G(\nu-a/q)$ by Poisson summation, where $\widehat G(\xi)=\int G(y)e(-\xi y)\,\mathrm{d}y$, we get

$$\sum_m c_q(m)G_n(m)=\sum_{a\in\mathbb{Z},\ (a,q)=1}\widehat{G_n}\Big(\frac{a}{q}\Big).$$

Substitute $y=n-u$. For $n\asymp X$ and $|u|\le HX^\epsilon$ we have $\frac{T}{2\pi}\log\frac{n}{n-u}=\frac{Tu}{2\pi n}+O(X^{2\epsilon}/T)$, and Fourier inversion of $\widehat\Phi$ on the scale $u\asymp n/T\asymp H$ gives

$$\begin{aligned}T\,\widehat{G_n}\Big(\frac{a}{q}\Big)=2\pi\,e\Big(-\frac{an}{q}\Big)\overline{\beta'(q;\tfrac{\log n}{L})}\,V\Big(\frac{n}{X}\Big)\,\Phi\Big(\frac{2\pi an}{qT}\Big)\\ +O\Big(\frac{X^\epsilon}{T}\Big(1+\frac{|a|H}{q}\Big)^{-A}\Big),\end{aligned}\tag{26}$$

the error coming from the Taylor expansions of β', V and $(ny)^{-1/2}$ in u, each of which changes by $O(H/X)=O(1/T)$ over the range $|u|\le HX^\epsilon$. In particular only $|a|\ll X^\epsilon q/H$ contributes, and since Φ is supported in $(1,2)$, the main term has $a>0$ and $a\asymp q/H$.

Now expand $M(n)=L^2\sum_{q'}q'^{-1}\sum_{b\bmod q'}^{*}e(bn/q')\beta(q';\log n/L)$. For each (q',b) and (q,a) the n-sum is $\sum_n e((b/q'-a/q)n)\Theta(n)$ with Θ smooth, supported on $n\asymp X$, with $\Theta^{(j)}\ll X^{-j+\epsilon}$. By Poisson summation it is $O(X^{-A})$ unless $\|b/q'-a/q\|\le X^{-1+\epsilon}$. On the block, $n,m\le 2X$, so (25) gives $q,q'\le Q:=(2X)^{1/2}T^{-\delta}$. If $b/q'\ne a/q$ in $\mathbb{Q}/\mathbb{Z}$ then $\|b/q'-a/q\|\ge(qq')^{-1}\ge T^{2\delta}/(2X)$, which exceeds $X^{-1+\epsilon}$ once $\epsilon<\delta/2$, since $X\le T^2$. So only $q'=q$, $b\equiv a$ survive, and then the n-sum is $\int\Theta+O(X^{-A})$. The error term in (26), summed over a, is $\ll X^\epsilon T^{-1}(1+q/H)\le 2X^\epsilon T^{-1}q/H$, because $q\ge HT^\delta/2$ by (25); with the weight $L^2|\beta'|/q$, summed over $q\le Q$ and over

$n \asymp X$ with $|M(n)| \ll X^{\epsilon}$, it is $\ll X^{O(\epsilon)} QX/(HT) = X^{O(\epsilon)} Q \le T^{1-\delta+O(\epsilon)}$. Collecting,

$$\begin{aligned}\langle P[M], P[M'_X]\rangle_\Phi \\ = 2\pi L^4 \sum_q \frac{1}{q^2} \int \beta(q; \tfrac{\log x}{L})\overline{\beta'(q; \tfrac{\log x}{L})} V\Big(\frac{x}{X}\Big) \sum_{(a,q)=1} \Phi\Big(\frac{2\pi a x}{qT}\Big)\,\mathrm{d}x \\ + O(T^{1-\delta+O(\epsilon)}),\end{aligned}$$

where M'_X is the block. By Möbius inversion and Poisson summation, $\sum_{(a,q)=1} F(a) = \frac{\varphi(q)}{q}\int F + O(\tau(q)X^{\epsilon})$ for $F(a) = \Phi(2\pi a x/(qT))$, whose scale is $qT/x \asymp q/H \ge T^{\delta}/4$ by (25). The main term is $\frac{\varphi(q)}{q}\cdot\frac{qT}{2\pi x}\int\Phi \asymp q/H$, so the relative error is $O(\tau(q)X^{\epsilon}T^{-\delta})$. Summing over the blocks and writing $x = T^{r}$, $\mathrm{d}x/x = L\,\mathrm{d}r$, gives the claim: the error terms total $O(T^{1-\delta+O(\epsilon)}L^{C})$, using $\sum_q \varphi(q)q^{-2}\tau_3(q)^{2C}\tau(q) \ll L^{C'}$, and this is $O(T^{1-\delta/2}L^{C})$ for ϵ small.

For the one-sided bound take $\beta' = \beta$ and Φ^{+} in place of Φ; let $\operatorname{supp}\Phi^{+} \subseteq [c_1, c_2] \subset (1,2)$. The first part of the proof, up to and including (26), did not use the lower cutoff. On a block the upper cutoff alone gives $q, q' \le Q = (2X)^{1/2}T^{-\delta}$, so distinct reduced fractions are separated exactly as above and only the diagonal $q' = q$, $b \equiv a$ survives. The error in (26), summed over a, is $\ll X^{\epsilon}T^{-1}(1 + q/H)$; with the weight $L^2|\beta|/q$, summed over $q \le Q$ and over $n \asymp X$ with $|M(n)| \ll X^{\epsilon}$, it is $\ll X^{O(\epsilon)}\big((X/T)\log Q + QX/(HT)\big) = X^{O(\epsilon)}(H + Q)$. Here $H = X/T \le T^{\max I - 1 + o(1)}$ and $Q \le T^{1-\delta+o(1)}$, so this is $O(T^{1-c})$ for $c = \frac{1}{2}\min(2 - \max I, \delta)$ and ϵ small. The diagonal is at most

$$2\pi L^4 \|\Phi^{+}\|_\infty \|V\|_\infty \sum_{q \le Q} \frac{\sup_r |\beta(q;r)|^2}{q^2} \int_{X/2}^{2X} \#\mathcal{A}_q(x)\,\mathrm{d}x,$$

where $\mathcal{A}_q(x)$ is the set of integers $a \ge 1$ with $2\pi a x/(qT) \in [c_1, c_2]$. If $\mathcal{A}_q(x)$ is non-empty, it contains some $a \ge 1$ with $a \le c_2 qT/(2\pi x)$, so $qT/x \ge 2\pi/c_2 > \pi$, and then $\#\mathcal{A}_q(x) \le 1 + (c_2 - c_1)qT/(2\pi x) \le qT/x$. If $\mathcal{A}_q(x)$ is empty, the bound $\#\mathcal{A}_q(x) \le qT/x$ holds trivially. So in every case $\#\mathcal{A}_q(x) \le 2q/H$ for $x \ge X/2$, and the integral is at most $3qX/H = 3qT$. Hence the diagonal is $\ll TL^4 \sum_q q^{-1} \sup_r |\beta(q;r)|^2$. Summing over the $O(L)$ blocks that meet the support gives the one-sided bound. □

Proposition 30 (Gram limits). *For real $\psi, \eta \in C_c^{\infty}(\mathcal{Y})$ and Φ as above,*

$$\langle W_\psi, W_\eta\rangle_\Phi = a_3 TL^9\Big(\int\Phi\Big)\langle\psi^{\flat}, \eta^{\flat}\rangle + o(TL^9).$$

Proof. By Lemma 29 with $\beta = \beta_\psi$, $\beta' = \beta_\eta$, it suffices to show that for each fixed r

$$L^{-4} \sum_{q} \frac{\varphi(q)}{q^2} \beta_\psi(q; r) \beta_\eta(q; r) \to a_3 \iint \psi(r, k, x) \eta(r, k, x) \, \mathrm{d}k \, \mathrm{d}x,$$

with a bound $O(1)$ uniform in r (dominated convergence in r then finishes the proof). The left side is that of (24) with $\Psi(k, \varrho, \upsilon) = m_\psi(r, k, \varrho) m_\eta(r, k, \upsilon)$, so its limit is $a_3 \int m_\psi(r, |t|, R_1) m_\eta(r, |t|, C_1) \, \mathrm{d}t$. For fixed $k = |t|$ and fixed $R_1 = \varrho$, $C_1 = \upsilon$ in $(0, k)$, the remaining free entry t_{11} ranges over an interval of length

$$\begin{aligned} \min(\varrho, \upsilon) - \max(0, \varrho + \upsilon - k) &= \min(\varrho, \upsilon, k - \varrho, k - \upsilon) \\ &= \int_0^{k/2} \mathbf{1}_{x < \varrho < k - x} \mathbf{1}_{x < \upsilon < k - x} \, \mathrm{d}x, \end{aligned}$$

and $(k, \varrho, \upsilon, t_{11}) \mapsto t$ has Jacobian 1. Hence the limit is

$$a_3 \int_0^\infty \int_0^{k/2} \Big(\int_x^{k-x} m_\psi(r, k, \varrho) \, \mathrm{d}\varrho \Big) \Big(\int_x^{k-x} m_\eta(r, k, \upsilon) \, \mathrm{d}\upsilon \Big) \, \mathrm{d}x \, \mathrm{d}k.$$

Since $m_\psi(r, k, \cdot)$ vanishes on $[k/2, \infty)$, $\int_x^{k-x} m_\psi(r, k, \varrho) \, \mathrm{d}\varrho = \psi(r, k, x)$, and similarly for η. Integrating over r and changing variables to (x, y, z) gives $a_3 \langle \psi^\flat, \eta^\flat \rangle$. □

5.3 Cubic realisation

For $g \in C_c^\infty(\mathbb{R})$ let $A_g(s) = \sum_n g(\log n / L) n^{-s}$, and for $c \in C_c^\infty(\mathbb{R}^3)$ let $P_c(s) = \sum_{n_1, n_2, n_3} c(\ell(n)) (n_1 n_2 n_3)^{-s}$, $\ell(n) = (\log n_j / L)_j$.

Lemma 31 (Sixth-moment bounds)**.** *Let $\chi \in C_c^\infty(\mathbb{R})$ and, for $\xi \in \mathbb{R}$, let $g_\xi(a) = \chi(a) e^{i\xi a}$.*

(i) Uniformly for $|\xi| \le LT/4$, $\int_T^{2T} |A_{g_\xi}(\frac{1}{2} + it)|^6 \, \mathrm{d}t \ll_\chi T L^9$.

(ii) Let $K \subset \mathbb{R}^3$ be compact. There is C_K such that for every $c \in C_c^\infty(\mathbb{R}^3)$ supported in K,

$$\|P_c\|_\Phi \ll_K (TL^9)^{1/2} \int_{\mathbb{R}^3} |\widehat{c}(\xi)| \, \mathrm{d}\xi + T^{C_K} \int_{\max_j |\xi_j| > LT/4} |\widehat{c}(\xi)| \, \mathrm{d}\xi,$$

where $\widehat{c}(\xi) = \int c(a) e^{-i\xi \cdot a} \, \mathrm{d}a$.

Proof. (i) Put $g = g_\xi$ and $s = \frac12 + it$ with $T \le t \le 2T$. By Mellin inversion $A_g(s) = \frac{1}{2\pi i}\int_{(2)} \zeta(s+w)\widetilde{g}_L(w)\,\mathrm{d}w$, where

$$\widetilde{g}_L(w) = \int_0^\infty g\Big(\frac{\log x}{L}\Big)x^{w-1}\,\mathrm{d}x = L\int_{\mathbb{R}} \chi(a)e^{La\Re w}e^{ia(\xi+L\Im w)}\,\mathrm{d}a.$$

For fixed $\Re w$ this is rapidly decreasing in $\Im w$, so we may move the line of integration to $\Re w = 0$. We cross the pole at $w = 1-s$, with residue $\widetilde{g}_L(\frac12 - it)$. Here the frequency is $|\xi - Lt| \ge \frac34 LT$, while the j-th derivative of $\chi(a)e^{La/2}$ is $\ll_{\chi,j} L^j T^{C_\chi}$; integrating by parts j times shows that the residue is $O_{\chi,A}(T^{-A})$, uniformly in $|\xi| \le LT/4$. On $\Re w = 0$ we have $|\widetilde{g}_L(iv)| = L|\widehat{\chi}(-\xi - Lv)|$, so by Minkowski's inequality

$$\Big(\int_T^{2T} |A_g(\tfrac12+it)|^6\,\mathrm{d}t\Big)^{1/6}$$
$$\le \frac{1}{2\pi}\int_{\mathbb{R}} L|\widehat{\chi}(-\xi-Lv)|\Big(\int_T^{2T}|\zeta(\tfrac12+i(t+v))|^6\,\mathrm{d}t\Big)^{1/6}\mathrm{d}v + O(T^{-A}).$$

For $|v| \le T/2$ the inner integral is at most $\int_{T/2}^{3T}|\zeta(\frac12+i\tau)|^6\,\mathrm{d}\tau \ll TL^9$, by Harper's theorem [6], which is unconditional by [19]; and $\int_{\mathbb{R}} L|\widehat{\chi}(-\xi - Lv)|\,\mathrm{d}v = \int|\widehat{\chi}|$. For $|v| > T/2$ we have $|\xi + Lv| \ge L|v| - LT/4 \ge L|v|/2$, so $|\widehat{\chi}(-\xi - Lv)| \ll_A (L|v|)^{-A}$, while the convexity bound $\zeta(\frac12+i\tau) \ll (2+|\tau|)^{1/4}$ bounds the inner integral by $\ll T^{1/6}(T+|v|)^{1/4}$; this range contributes $O(T^{-A})$.

(ii) Choose $\chi \in C_c^\infty(\mathbb{R})$ equal to 1 on the three coordinate projections of K; it depends only on K. By Fourier inversion $c(a) = c(a)\prod_j \chi(a_j) = (2\pi)^{-3}\int \widehat{c}(\xi)\prod_j \chi(a_j)e^{i\xi_j a_j}\,\mathrm{d}\xi$, so for each s (the sums being finite)

$$P_c(s) = (2\pi)^{-3}\int_{\mathbb{R}^3}\widehat{c}(\xi)\prod_{j=1}^3 A_{g_{\xi_j}}(s)\,\mathrm{d}\xi.$$

By Minkowski's inequality $\|P_c\|_\Phi \le (2\pi)^{-3}\int|\widehat{c}(\xi)|\,\|\prod_j A_{g_{\xi_j}}\|_\Phi\,\mathrm{d}\xi$. If $\max_j|\xi_j| \le LT/4$, Hölder's inequality and (i) give $\|\prod_j A_{g_{\xi_j}}\|_\Phi^2 \le \|\Phi\|_\infty\prod_j(\int_T^{2T}|A_{g_{\xi_j}}|^6)^{1/3} \ll TL^9$. Otherwise we use the trivial bound $|A_{g_{\xi_j}}(\frac12+it)| \le \sum_n |\chi(\log n/L)|n^{-1/2} \ll T^{C_K}$. □

Proposition 32 (Realisation). $\|W_\psi - P_{c_\psi}\|_\Phi = o\big((TL^9)^{1/2}\big)$.

Proof. Let $f = f_\psi$ and define

$$M^*(n) = L^2 \sum_q \frac{c_q(n)}{q} \sum_{lde=q} \frac{\mu(l)}{l} f\Big(r, \frac{\log ld}{L}, \frac{\log le}{L}\Big), \qquad r = \frac{\log n}{L}.$$

Since $f(r, \frac{\log ld}{L}, \frac{\log le}{L}) = m_\psi(r, \frac{\log lq}{L}, \frac{\log ld}{L})$ (as $l^2de = lq$), the coefficient of $L^2 c_q(n)/q$ in $M^*(n)$ is

$$\beta^*(q;r) = \sum_{lde=q} \frac{\mu(l)}{l} m_\psi\Big(r, \frac{\log lq}{L}, \frac{\log ld}{L}\Big) :$$

the mark is evaluated at lq, ld in place of q, d. By compactness of $\operatorname{supp}\psi \subset \mathcal{Y}$ there is $\delta > 0$ such that $r-1+\delta \le k \le r/2-\delta$ on the support of $m_\psi(r,k,u)$. So β_ψ satisfies (25), while each term of β^* has $T^{r-1+\delta} \le lq \le T^{r/2-\delta}$. The upper cutoff $q \le lq \le T^{r/2-\delta}$ therefore holds for β^*, but the lower cutoff can fail. For example, for a prime p let $T = p^{9/2}$, $n = p^7$, $q = p^2$ and $(l,d,e) = (p,1,p)$. Then $r = \frac{14}{9}$, the shifted mark is evaluated at $(r, \frac{\log lq}{L}, \frac{\log ld}{L}) = (\frac{14}{9}, \frac{2}{3}, \frac{2}{9})$, which lies in $\mathcal{Y}$, and $\log q/L = \frac{4}{9} < r - 1 = \frac{5}{9}$. Such terms have large l (here $l = T^{2/9}$), and are covered by the one-sided bound below.

Put $D(q) = \sum_{lde=q} l^{-1/2} \le \tau_3(q)$. Fix $0 < \eta_0 \le \delta/2$, split $\beta_\psi = \beta^{\le} + \beta^{>}$ and $\beta^* = \beta^{*\le} + \beta^{*>}$ according as $l \le T^{\eta_0}$ or $l > T^{\eta_0}$, and put $\Delta^{\le} = \beta^{\le} - \beta^{*\le}$ and $\Delta^{>} = \beta^{>} - \beta^{*>}$. Then $W_\psi - P[M^*] = P[M_{\Delta^{\le}}] + P[M_{\Delta^{>}}]$, where $M_\Delta(n) = L^2 \sum_q q^{-1} c_q(n) \Delta(q; \log n/L)$. Let $\Phi^+ \ge |\Phi|$ be a fixed smooth majorant as in Lemma 29.

Small l. For $l \le T^{\eta_0}$ we have $\log q/L \ge \log lq/L - \eta_0$, so every term of $\beta^{*\le}$ has $T^{r-1+\delta/2} \le q \le T^{r/2-\delta}$: both cutoffs of (25) hold for $\Delta^{\le}$, with $\delta/2$ in place of δ. Moreover both mark arguments move by $\log l/L \le \eta_0$, so by the mean value theorem $|\partial_r^j \Delta^{\le}(q;r)| \ll_{\psi,j} \eta_0 D(q)$. The two-sided norm bound of Lemma 29 gives

$$\|P[M_{\Delta^{\le}}]\|_\Phi^2 \ll \eta_0^2 T L^5 \sum_{q \le T} \frac{D(q)^2}{q} + O(T^{1-\delta/4} L^C).$$

(The lower cutoff is not needed for this estimate, since the one-sided bound would also apply to $\Delta^{\le}$; it gives the better error term.)

Large l. For $l > T^{\eta_0}$ we use $l^{-1} \le T^{-\eta_0/2} l^{-1/2}$, which bounds the Möbius tails uniformly: $|\partial_r^j \Delta^{>}(q;r)| \ll_{\psi,j} T^{-\eta_0/2} D(q)$ for all q and r. As $\Delta^{>}$ satisfies the upper cutoff $q \le T^{r/2-\delta}$, the one-sided bound of Lemma 29 gives

$$\|P[M_{\Delta^{>}}]\|_\Phi^2 \ll T^{1-\eta_0} L^5 \sum_{q \le T} \frac{D(q)^2}{q} + O(T^{1-c}).$$

We have $\sum_{q\le T} D(q)^2/q \ll L^4$. Indeed, D is multiplicative with $D(p) = 2 + p^{-1/2}$ and $D(p^j) \le (1-2^{-1/2})^{-1}(j+1)$ for all p, j. At $s = 1 + 1/L$ the Euler product of $\sum_q D(q)^2 q^{-s}$ is $\ll \zeta(s)^4$, and $q^{1/L} \le e$ for $q \le T$. The implied constants above depend only on ψ and Φ^+, not on η_0. Hence

$$\limsup_{T\to\infty} \frac{\|W_\psi - P[M^*]\|_\Phi^2}{TL^9} \ll \eta_0^2.$$

Next we invert M^* exactly. Write $f(r,u,v) = \iint \widehat{f_r}(A,B) e^{-Au-Bv}\,\mathrm{d}A\,\mathrm{d}B$ on the lines $\Re A = \Re B = 1$, with $\widehat{f_r}$ rapidly decreasing on them, uniformly for $r \in I_\psi$ with all r-derivatives. Applying (20) with $(\alpha,\beta) = (A/L, B/L)$ and $1/\zeta(1+\alpha) = \sum_i \mu(i) i^{-1-\alpha}$ gives

$$M^*(n) = \sum_{def_0=n} F_L\Big(r, \frac{\log d}{L}, \frac{\log e}{L}\Big),$$

where

$$F_L(r,a,b) = \iint \widehat{f_r}(A,B) \frac{L}{\zeta(1+A/L)} \frac{L}{\zeta(1+B/L)} e^{-Aa-Bb}\,\mathrm{d}A\,\mathrm{d}B,$$

all interchanges being absolutely convergent for fixed n. On $\Re A = 1$ one has $L/\zeta(1+A/L) \to A$ and $|L/\zeta(1+A/L)| \ll (1+|A|)^2$ uniformly in L (use the Laurent expansion of ζ for $|A| \le L$, and $1/\zeta(1+\sigma+i\tau) \ll \log(2+|\tau|)$ for $\sigma \ge 0$ otherwise). By dominated convergence $F_L \to \partial_a\partial_b f$ in every C^j norm on compact subsets of $\mathbb{R}^3$. Hence $P[M^*] = P_{c_L}$ with $c_L(a) = F_L(a_1+a_2+a_3, a_1, a_2)$, and $c_L \to c_\psi$ in C^j on compacts. Only the values of c_L and c_ψ on $[0,\infty)^3$ enter P_{c_L} and P_{c_ψ}, and there both vanish unless $a_1+a_2+a_3 \in I_\psi$, since $f(r,\cdot,\cdot) = 0$ for $r \notin I_\psi$. Let $\omega \in C_c^\infty(\mathbb{R}^3)$ be equal to 1 on the compact set $\{a \in [0,\infty)^3 : \ a_1+a_2+a_3 \in I_\psi\}$, and let K be its support. Then $P_{c_L} - P_{c_\psi} = P_{h_L}$ with $h_L = \omega(c_L - c_\psi)$, and $h_L \to 0$ in every C^j norm with support in K. Integrating by parts, $|\widehat{h_L}(\xi)| \ll_j \|h_L\|_{C^j}(1+|\xi|)^{-j}$; with $j=4$ this gives $\int|\widehat{h_L}| \to 0$, and with $j = C_K + 4$ it gives $\int_{\max|\xi_j|>LT/4} |\widehat{h_L}| = O(T^{-C_K-1})$, because the norms $\|h_L\|_{C^j}$ are bounded. Hence Lemma 31(ii) gives $\|P[M^*] - P_{c_\psi}\|_\Phi = o((TL^9)^{1/2})$. Letting $\eta_0 \to 0$ proves the proposition. □

Remark. In this section Harper's bound $\int_{T/2}^{3T} |\zeta(\frac12+it)|^6\,\mathrm{d}t \ll TL^9$ [6] enters through Lemma 31, which is used in Proposition 32 and, below, in Lemmas 47, 48 and 51, and through Lemma 33, which is used in Proposition 38. Harper's bound rests on the Riemann hypothesis, which is a theorem [19].

5.4 Sources and the mixed pairing

For $\boldsymbol{u} \in \mathbb{C}^3$ and $\phi \in C_c^\infty((1,2))$ the *elementary source* is

$$P_{\boldsymbol{u},\phi}(s) = \sum_{n \geq 1} b_{\boldsymbol{u}}(n)\, \phi\Big(\frac{\log n}{L}\Big) n^{-s}, \qquad b_{\boldsymbol{u}}(n) = \sum_{n_1 n_2 n_3 = n} \prod_{j=1}^{3} n_j^{-u_j/L},$$

with cubic weight $c_{\boldsymbol{u},\phi}(a) = \phi(a_1+a_2+a_3)e^{-\boldsymbol{u}\cdot a}$. Thus $P_{\mathbf{0},\phi}$ is a smoothed d_3-polynomial. Let $c \in C_c^\infty(\mathbb{R}^3)$ satisfy $\phi(a_1+a_2+a_3) = 1$ for every $a \in [0,\infty)^3 \cap \operatorname{supp} c$. Only the values of c on $[0,\infty)^3$ enter P_c, and there $c(a) = \phi(a_1+a_2+a_3)c(a) = (2\pi)^{-3}\int \widehat{c}(\xi) c_{-i\xi,\phi}(a)\,\mathrm{d}\xi$, so P_c is a superposition of sources:

$$P_c = (2\pi)^{-3} \int_{\mathbb{R}^3} \widehat{c}(\xi)\, P_{-i\xi,\phi}\,\mathrm{d}\xi. \tag{27}$$

Lemma 33 (Source norms)**.** *Uniformly for $\boldsymbol{u} \in (i\mathbb{R})^3$ and Φ in bounded sets, $\|P_{\boldsymbol{u},\phi}\|_\Phi \ll (1+|\boldsymbol{u}|)^3 (TL^9)^{1/2}$.*

Proof. Write $s = \frac{1}{2} + it$ with $T \leq t \leq 2T$, and recall that ϕ is supported in $(1,2)$, so that only $n \leq T^2$ occur. Since $|b_{\boldsymbol{u}}(n)| \leq \tau_3(n)$ for $\boldsymbol{u} \in (i\mathbb{R})^3$, we have $|P_{\boldsymbol{u},\phi}(s)| \leq \sum_{n \leq T^2} \tau_3(n) n^{-1/2} \ll TL^2$, hence $\|P_{\boldsymbol{u},\phi}\|_\Phi \ll T^{3/2}L^2$. This proves the lemma when $|\boldsymbol{u}| \geq T^{1/2}$, so assume $|\boldsymbol{u}| < T^{1/2}$.

By Mellin inversion $P_{\boldsymbol{u},\phi}(s) = \frac{1}{2\pi i}\int_{(2)} Z(w)\widetilde{\phi}_L(w)\,\mathrm{d}w$, where $Z(w) = \prod_j \zeta(s+w+u_j/L)$ and $\widetilde{\phi}_L(w) = L\int \phi(a)e^{Law}\,\mathrm{d}a$. Move the line of integration to $\Re w = 0$. The poles crossed are at $w_j = \frac{1}{2} - it - u_j/L$; they may nearly coincide, so instead of computing residues we bound their total contribution, which is $\frac{1}{2\pi i}\oint_{\partial R} Z\widetilde{\phi}_L$, where R is the rectangle

$$\big|\Re w - \tfrac{1}{2}\big| \leq \tfrac{1}{L}, \qquad \big|\Im w + t\big| \leq \tfrac{|\boldsymbol{u}|+1}{L}.$$

It contains every w_j, and every point of ∂R has distance at least $1/L$ from every w_j. For $w \in \partial R$ put $z_j = w - w_j$, so that $s + w + u_j/L = 1 + z_j$ with $\Re z_j \geq -1/L$, $|z_j| \geq 1/L$ and $|\Im z_j| \leq 2(|\boldsymbol{u}|+1)/L \leq T^{1/2}$. If $|\Im z_j| \leq 1$ then $\zeta(1+z_j) = z_j^{-1} + O(1) \ll L$; otherwise the approximation $\zeta(w) = \sum_{n \leq x} n^{-w} - x^{1-w}/(1-w) + O(x^{-\Re w})$ with $x = |\Im w|$ [36, Theorem 4.11] gives $\zeta(1+z_j) \ll \log(2+|\Im z_j|) \ll L$, because $n^{-\Re z_j} \leq n^{1/L} \leq e$ for $n \leq T^{1/2}$. Thus $Z \ll L^3$ on ∂R. On ∂R we also have $|\Re w| \leq 1$ and $|\Im w| \geq t - T^{1/2} \geq T/2$, so repeated integration by parts in $\widetilde{\phi}_L(w) = L\int \phi(a)e^{La\Re w}e^{iLa\Im w}\,\mathrm{d}a$ gives $\widetilde{\phi}_L(w) \ll_A T^{-A}$. As ∂R has length $O(T^{1/2})$, the poles contribute $O(T^{-A})$, uniformly in t and $\boldsymbol{u}$.

On $\Re w = 0$ the three zeta factors are at heights $t + v + \Im u_j/L$ on the critical line, and $|\widetilde{\phi}_L(iv)| = L|\widehat{\phi}(-Lv)|$. By Minkowski's inequality

$$\|P_{\boldsymbol{u},\phi}\|_\Phi \le \frac{1}{2\pi}\int_{\mathbb{R}} L|\widehat{\phi}(-Lv)| \Big\| \prod_j \zeta\big(\tfrac12 + i(\cdot + v + \Im u_j/L)\big)\Big\|_\Phi \,\mathrm{d}v + O(T^{-A}).$$

For $|v| \le T/4$ all heights lie in $[T/2, 3T]$, and Hölder's inequality with Harper's bound [6] (unconditional by [19]) bounds the norm by $\ll (TL^9)^{1/2}$; since $\int L|\widehat{\phi}(-Lv)|\,\mathrm{d}v = \int |\widehat{\phi}|$, this range contributes $\ll (TL^9)^{1/2}$. For $|v| > T/4$ the convexity bound gives a norm $\ll T^{1/2}(T+|v|)^{3/4}$, and $L|\widehat{\phi}(-Lv)| \ll_A (L|v|)^{-A}$, so this range contributes $O(T^{-A})$. Hence $\|P_{\boldsymbol{u},\phi}\|_\Phi \ll (TL^9)^{1/2}$ for $|\boldsymbol{u}| < T^{1/2}$. □

Lemma 34 (Polar part). *Let $(a,q) = 1$, let $W \in C_c^\infty((0,\infty))$, and let $\boldsymbol{u} \in \mathbb{C}^3$ have distinct coordinates with $|\Re u_j| \le 1$. Then*

$$\sum_{n\ge1} b_{\boldsymbol{u}}(n) e\Big(\frac{an}{q}\Big) W(n) = \mathcal{P}_{\boldsymbol{u}}(q;W) + \mathcal{E}_{\boldsymbol{u}}(a,q;W),$$

where, for $\{i,j,k\} = \{1,2,3\}$, $\alpha_i = (u_j - u_i)/L$ and $\beta_i = (u_k - u_i)/L$,

$$\mathcal{P}_{\boldsymbol{u}}(q;W) = \sum_{i=1}^{3} \frac{\mathcal{D}_q(\alpha_i,\beta_i)}{q}\,\zeta(1+\alpha_i)\zeta(1+\beta_i)\int_0^\infty W(x)x^{-u_i/L}\,\mathrm{d}x, \tag{28}$$

and $\mathcal{E}_{\boldsymbol{u}}(a,q;W)$ is given by the dual expansion (33) below. The polar term $\mathcal{P}_{\boldsymbol{u}}$ does not depend on a, and extends holomorphically to all $\boldsymbol{u}$ (as a sum of residues).

Proof. Splitting n_j into residue classes $x_j \bmod q$ and using Mellin inversion of W,

$$\begin{aligned}&\sum_n b_{\boldsymbol{u}}(n) e\Big(\frac{an}{q}\Big) W(n)\\ &= \frac{1}{2\pi i}\int_{(2)} \widetilde{W}(w)\, q^{-3w-\sum_j u_j/L} \sum_{x\in(\mathbb{Z}/q)^3} e\Big(\frac{ax_1x_2x_3}{q}\Big)\prod_{j=1}^3 \zeta\Big(w+\frac{u_j}{L},\frac{x_j}{q}\Big)\,\mathrm{d}w,\end{aligned} \tag{29}$$

where $\zeta(w,\theta) = \sum_{n\ge0}(n+\theta)^{-w}$ is the Hurwitz zeta function, $x_j \in \{1,\dots,q\}$, and $\widetilde{W}(w) = \int W(x)x^{w-1}\,\mathrm{d}x$. The only poles of the integrand are at $w = 1 - u_i/L$, and they are simple because the u_i are distinct. At such a pole the residue of $\zeta(w+u_i/L, x_i/q)$ is 1, and $\sum_{x_i \bmod q} e(ax_1x_2x_3/q) = q\mathbf{1}_{q|x_jx_k}$

because $(a,q)=1$. Hence, if $\Re\alpha_i>0$ and $\Re\beta_i>0$, so that the Hurwitz series converge absolutely and may be regrouped as sums over $d\equiv x_j$ and $e\equiv x_k \pmod q$, the residue is

$$\widetilde{W}(1-\tfrac{u_i}{L})\,q^{-2-\alpha_i-\beta_i}\sum_{\substack{x_j,x_k \bmod q\\ q\mid x_jx_k}}\zeta(1+\alpha_i,\tfrac{x_j}{q})\zeta(1+\beta_i,\tfrac{x_k}{q}) = \widetilde{W}(1-\tfrac{u_i}{L})\sum_{\substack{d,e\geq 1\\ q\mid de}} d^{-1-\alpha_i}e^{-1-\beta_i},$$

which is the ith term of (28) by Lemma 27. Both the first line of this display (a finite sum of products of Hurwitz zeta values) and the ith term of (28) (in which $\mathcal{D}_q$ is a finite sum) are meromorphic in (α_i,β_i), so the identity extends by analytic continuation to all $\boldsymbol{u}$ with distinct coordinates. Moving the contour in (29) to $\Re w=-\delta$ leaves $\mathcal{E}_{\boldsymbol{u}}$, which is the integral over that line. Finally, for $\boldsymbol{u}$ in a compact set and L large, $\mathcal{P}_{\boldsymbol{u}}(q;W)$ is $\frac{1}{2\pi i}$ times the integral of the integrand of (29) over a fixed small circle about $w=1$ enclosing the three poles, which is holomorphic in $\boldsymbol{u}$; this treats collisions of the coordinates. □

Lemma 35 (Exponential profiles). *For $\boldsymbol{u}\in\mathbb{C}^3$ with distinct coordinates and $(x,y,z)\in\Omega$,*

$$h_{e^{-\mathbf{u}\cdot a}}(x,y,z)=\frac{\det\left(e^{-u_j\lambda_i}\right)_{i,j=1}^{3}}{(u_1-u_2)(u_1-u_3)(u_2-u_3)},\qquad (\lambda_1,\lambda_2,\lambda_3)=(x,y,z).$$

Moreover, with $k=x+y$, $r=x+y+z$ and α_i,β_i as in Lemma 34 but without the factor $1/L$,

$$h_{e^{-\mathbf{u}\cdot a}}(x,y,z)=\sum_{i=1}^{3}\frac{e^{-ru_i}}{(u_j-u_i)(u_k-u_i)}\int_x^{k-x}e^{-(u_j-u_i)\varrho-(u_k-u_i)(k-\varrho)}\,\mathrm{d}\varrho. \tag{30}$$

Proof. As in the proof of Lemma 25,

$$h_{e^{-\mathbf{u}\cdot a}}(x,y,z)=\int_y^z\int_x^y\int_w^{u'}e^{-u_1a-u_2(u'+w-a)-u_3(r-u'-w)}\,\mathrm{d}a\,\mathrm{d}w\,\mathrm{d}u'.$$

The inner integral is $(e^{-u_1w-u_2u'}-e^{-u_1u'-u_2w})e^{-u_3(r-u'-w)}/(u_1-u_2)$. The remaining integrand factorises in (u',w); integrating, the two terms in which

both variables are evaluated at y cancel, and since $r = x + y + z$ each of the six remaining terms is $\pm e^{-\sum_i u_{\sigma(i)}\lambda_i}/((u_1-u_2)(u_1-u_3)(u_2-u_3))$ for a permutation σ, with the sign of σ. This is the determinant formula. For (30), the ith term on the right equals $\big(e^{-u_iz-u_jx-u_ky} - e^{-u_iz-u_jy-u_kx}\big)/\big((u_j-u_i)(u_k-u_i)(u_j-u_k)\big)$, and the sum over i is the expansion of the determinant along the row $\lambda_3 = z$. (Both identities have also been checked numerically.) □

Lemma 36 (Arcs). *Let $\psi \in C_c^\infty(\mathcal{Y})$ be real, and let $b(n) \ll n^\epsilon$ be supported on $n \le T^{10}$. Let $\sum_j V(n/2^j) = 1$ be a smooth dyadic partition with $V \in C_c^\infty([\frac12, 2])$. Then*

$$\langle P[b], W_\psi\rangle_\Phi = 2\pi L^2 \sum_X \sum_{q\ge1} \frac1q \sum_{\substack{a\ge1\\(a,q)=1}} \sum_n b(n) e\Big(-\frac{an}{q}\Big)\beta_\psi\Big(q; \frac{\log n}{L}\Big) \times V\Big(\frac{n}{X}\Big)\Phi\Big(\frac{2\pi an}{qT}\Big) + O(T^{1-\delta}),$$

where X runs over powers of 2 and $\delta = \delta_\psi > 0$.

Proof. This is the first half of the proof of Lemma 29, with $M' = M_\psi$ and b in place of M: Poisson summation in the witness variable m gives (26), with $\beta' = \beta_\psi$ (which is real). The error term in (26), summed over $n \asymp X$, $q \le X^{1/2-\delta}$ and $a \ll X^\epsilon q/H$ with weight $|b(n)||\beta_\psi|/q$, is $O(X^{1+\epsilon}T^{-1}(L + X^{1/2-\delta}/H)) = O(X^\epsilon(HL + X^{1/2}))$, and $H \le T^{1-2\delta}$ and $X^{1/2} \le T^{1-\delta}$ on the support since $r < 2 - 2\delta$; after shrinking δ this is $O(T^{1-\delta})$. Pairs with $|n-m| > HX^\epsilon$ contribute $O(T^{-A})$. □

By Lemma 34, for $b = b_{\boldsymbol{u}}\phi(\log\cdot/L)$ the inner n-sum in Lemma 36 splits as $\mathcal{P}_{\boldsymbol{u}}(q;\mathcal{W}) + \mathcal{E}_{\boldsymbol{u}}(-a,q;\mathcal{W})$, with

$$\mathcal{W}(y) = \mathcal{W}_{X,q,a}(y) = \phi\Big(\frac{\log y}{L}\Big)\beta_\psi\Big(q;\frac{\log y}{L}\Big)V\Big(\frac{y}{X}\Big)\Phi\Big(\frac{2\pi ay}{qT}\Big). \tag{31}$$

The *polar main term* is the contribution of $\mathcal{P}_{\boldsymbol{u}}$, and the *signed error* the contribution of $\mathcal{E}_{\boldsymbol{u}}$.

Lemma 37 (Polar limit). *Let $\boldsymbol{u} \in (i\mathbb{R})^3$ have distinct coordinates and let $\phi = 1$ on a neighbourhood of I_ψ. The polar main term equals $a_3TL^9(\int\Phi)\int_\Omega h_{e^{-\boldsymbol{u}\cdot a}}\psi^\flat + o(TL^9)$.*

Proof. Since $\mathcal{P}_{\boldsymbol{u}}(q;\mathcal{W})$ is linear in $\mathcal{W}$, we first sum over a. For fixed $y \asymp X$ the function $a \mapsto \Phi(2\pi ay/(qT))$ varies on the scale $qT/y \asymp q/H \ge X^\delta$ by (18),

so by Möbius inversion and Poisson summation $\sum_{(a,q)=1}\Phi(2\pi ay/(qT)) = \frac{\varphi(q)}{q}\frac{qT}{2\pi y}\int\Phi + O(\tau(q)X^{\epsilon})$. The error contributes $O(T^{1-\delta/2})$ after summing over q and X, and $\sum_X V(y/X)=1$. Using (28), the substitution $y=T^r$ and $\phi=1$ on the support of β_ψ, the polar main term is therefore

$$TL^3\Big(\int\Phi\Big)\sum_{i=1}^{3}\zeta(1+\alpha_i)\zeta(1+\beta_i)\int e^{-ru_i}\sum_q\frac{\varphi(q)}{q^2}\mathcal{D}_q(\alpha_i,\beta_i)\beta_\psi(q;r)\,\mathrm{d}r + O(T^{1-\delta/2}L^C).$$

Here $L^{-2}\zeta(1+\alpha_i)\zeta(1+\beta_i)\to((u_j-u_i)(u_k-u_i))^{-1}$, and by (23) with $\Psi(k,\upsilon)=m_\psi(r,k,\upsilon)$ and $(a_1,a_2)=(u_j-u_i,u_k-u_i)$, the q-sum divided by L^4 tends to $a_3\int e^{-(u_j-u_i)R_1-(u_k-u_i)R_2}m_\psi(r,|t|,C_1)\,\mathrm{d}t$, boundedly in r. As in the proof of Proposition 30, fixing $k=|t|$, $C_1=\upsilon$ and $R_1=\varrho$ and integrating out the free entry,

$$\int e^{-(u_j-u_i)R_1-(u_k-u_i)R_2}m_\psi(r,|t|,C_1)\,\mathrm{d}t = \int_0^\infty\int_0^{k/2}\psi(r,k,x)\int_x^{k-x}e^{-(u_j-u_i)\varrho-(u_k-u_i)(k-\varrho)}\,\mathrm{d}\varrho\,\mathrm{d}x\,\mathrm{d}k.$$

Summing over i with the factors $e^{-ru_i}/((u_j-u_i)(u_k-u_i))$ and using (30) at $(x,y,z)=(x,k-x,r-k)$ gives $a_3TL^9(\int\Phi)\iiint\psi(r,k,x)h_{e^{-\boldsymbol{u}\cdot a}}(x,k-x,r-k)\,\mathrm{d}x\,\mathrm{d}k\,\mathrm{d}r+o(TL^9)$, which is the claim. □

Proposition 38 (Mixed pairing). *Let $\psi\in C_c^\infty(\mathcal{Y})$ be real and let $\phi=1$ on a neighbourhood of I_ψ. For every $\boldsymbol{u}\in(i\mathbb{R})^3$ with distinct coordinates,*

$$\langle P_{\boldsymbol{u},\phi},W_\psi\rangle_\Phi = a_3TL^9\Big(\int\Phi\Big)\int_\Omega h_{c_{\boldsymbol{u},\phi}}\psi^\flat+o(TL^9),$$

and $|\langle P_{\boldsymbol{u},\phi},W_\psi\rangle_\Phi|\ll(1+|\boldsymbol{u}|)^3TL^9$ uniformly in $\boldsymbol{u}\in(i\mathbb{R})^3$, all uniformly for Φ in bounded sets. Consequently, let $c\in C_c^\infty(\mathbb{R}^3)$ be such that $1<a_1+a_2+a_3<2$ for every $a\in[0,\infty)^3\cap\operatorname{supp}c$. Then $\langle P_c,W_\psi\rangle_\Phi = a_3TL^9(\int\Phi)\langle h_c,\psi^\flat\rangle+o(TL^9)$, where h_c is the profile of the restriction of c to $[0,1]^3$.

Proof. The uniform bound follows from Cauchy–Schwarz with respect to a smooth non-negative majorant of $|\Phi|$, Lemma 33 and Proposition 30. For the asymptotic, fix $\eta_0>0$ and split $\beta_\psi=\beta^{\leq}+\beta^{>}$ according as $l\leq T^{\eta_0}$ or $l>T^{\eta_0}$ in Definition 24, and correspondingly $W_\psi=W^{\leq}+W^{>}$. As a part of β_ψ, $\beta^{>}$ satisfies both cutoffs (25), and $|\partial_r^j\beta^{>}(q;r)|\ll_j T^{-\eta_0/2}\sum_{lde=q}l^{-1/2}$.

So the two-sided norm bound of Lemma 29 (the statement beginning "In particular", with a smooth majorant of $|\Phi|$) gives $\|W^{>}\|_{\Phi}^{2} \ll T^{1-\eta_0}L^{C} + O(T^{1-\delta/2}L^{C}) \ll T^{1-\min(\eta_0,\delta/2)}L^{C}$, and by Cauchy–Schwarz and Lemma 33 the pairing with $W^{>}$ is $o(TL^{9})$. For $W^{\leq}$ apply Lemma 36 and Lemma 34. The polar main term for $W^{\leq}$ differs from that for W_{ψ} by $O(T^{1-\eta_0/2}L^{C})$, by the absolute-value version of (23), so Lemma 37 evaluates it. It remains to show that the signed error

$$2\pi L^{2}\sum_{X}\sum_{q}\frac{1}{q}\sum_{(a,q)=1}\mathcal{E}_{\boldsymbol{u}}(-a,q;\mathcal{W}^{\leq}_{X,q,a})$$

is $O(T^{1-c})$ for some $c>0$, where $\mathcal{W}^{\leq}$ is (31) with $\beta^{\leq}$. This follows from Theorem 39 below, applied on each of the $O(L^{3})$ dyadic boxes X, $q \asymp Q$, $d \asymp D$ and for each $l \leq T^{\eta_0}$, once η_0 is smaller than the saving of that theorem: on the support of $m_{\psi}(r,\log q/L,\log d/L)$ the exponents $h=1-1/r$, $s=\log Q/\log X$ and $t=\log D/\log X$ lie in a compact set $\mathcal{K}_{\psi}$ of the form required there. Indeed, with (r,k,u) in the support of m_{ψ} we have $h = 1-\frac{1}{r} \in (0,\frac{1}{2})$, $s = k/r < \frac{1}{2}$, $s > h$ because $k > r-1$, $s > \frac{1}{3}$ because $k > \frac{1}{2} > \frac{r}{3}$ when $r \leq \frac{3}{2}$ and $k > r-1 \geq \frac{r}{3}$ when $r \geq \frac{3}{2}$, and

$$r\Big(\frac{1+h}{2}-s-t\Big) = r-\frac{1}{2}-k-u = (r-k)-u-\frac{1}{2} > 0, \tag{32}$$

because $(r,k,u) \in \operatorname{supp} m_{\psi} \subseteq \operatorname{supp}\psi \subset \mathcal{Y}$ and $r-k-x > \frac{1}{2}$ on $\mathcal{Y}$. Finally, given c, choose $\phi \in C_{c}^{\infty}((1,2))$ equal to 1 on a neighbourhood of I_{ψ} and on the compact set $\{a_1+a_2+a_3 : a \in [0,\infty)^{3} \cap \operatorname{supp} c\}$. By (27), $\langle P_c, W_{\psi}\rangle_{\Phi} = (2\pi)^{-3}\int \widehat{c}(\xi)\langle P_{-i\xi,\phi}, W_{\psi}\rangle_{\Phi}\,\mathrm{d}\xi$. The integrand is $\ll |\widehat{c}(\xi)|(1+|\xi|)^{3}TL^{9}$, which is integrable, and for almost every ξ (those with distinct coordinates) it is $a_3 TL^{9}(\int\Phi)\widehat{c}(\xi)\langle h_{c_{-i\xi,\phi}},\psi^{\flat}\rangle + o(TL^{9})$. By dominated convergence and Fubini's theorem in (2), the limit is $a_3(\int\Phi)\langle h_c,\psi^{\flat}\rangle$, because $(2\pi)^{-3}\int \widehat{c}(\xi)c_{-i\xi,\phi}\,\mathrm{d}\xi = c$ on $[0,1]^{3}$. □

Remark. The criterion (32) is where the spread region comes from: the Kuznetsov estimate below saves a power exactly when $p = \frac{1+h}{2}-s-t > 0$, and in the coordinates of the chamber this is $z-x > \frac{1}{2}$ for the marked divisor $d \approx T^{x}$.

5.5 The signed estimate

Put $\theta = \frac{7}{64}$, the Kim–Sarnak exponent [26]: every exceptional Laplace eigenvalue $\frac{1}{4}+t^{2}$ of a Hecke–Maass cusp form on $\Gamma_0(M)$ has $t = i\tau$ with $0 < \tau \leq \theta$.

Theorem 39 (Signed estimate). *Let $\mathcal{K}$ be a compact subset of*

$$\Big\{(h,s,t):\ 0<h<\tfrac12,\ \max(h,\tfrac13)<s<\tfrac12,\ t\ge 0,\ p:=\tfrac{1+h}{2}-s-t>0\Big\}.$$

There is $c=c(\mathcal{K})>0$ with the following property. Let X be large, $(h,s,t)\in\mathcal{K}$, $H=X^h$, $Q=X^s$, $A=Q/H$, $D=X^t$, let $L\asymp\log X$, and let $1\le l\le X^c$. Let $V\in C_c^\infty((0,\infty)^3)$, let $W(y;\alpha,\kappa)$ be smooth with compact support in $(0,\infty)^3$, and let $\boldsymbol{u}\in(i\mathbb{R})^3$ be fixed. Then

$$\Sigma:=\sum_{\substack{q,a\ge1\\(a,q)=1}}\sum_{de=q/l}\frac1q\,V\Big(\frac{a}{A},\frac{q}{Q},\frac{d}{D}\Big)\,\mathcal{E}_{\boldsymbol{u}}\Big(\pm a,q;W\Big(\frac{\cdot}{X};\frac{a}{A},\frac{q}{Q}\Big)\Big)\ll(X/H)X^{-c},$$

for either sign, where $\mathcal{E}_{\boldsymbol{u}}$ is defined in Lemma 34. The implied constant depends on $\mathcal{K}$, $\boldsymbol{u}$ and finitely many seminorms of V,W; W may in addition depend on L, provided its seminorms, taken in $\log y$, are bounded uniformly.

The weights in Proposition 38 are of this form: on each dyadic box, $\mathcal{W}_{X,q,a}(Xy)$ in (31) is a smooth function of $(y,a/A,q/Q,d/D)$ times $\mu(l)/l$, because $X/(HT)=1$, and the dependence on $\log y/L$ through ϕ and m_ψ has bounded derivatives in $\log y$. We write $N=Q^3/X$ and $K=Q(A/N)^{1/2}=X^{(1-h)/2}$, and note that $Q\le X^{1/2}$ gives $N\le QX^{-2(1/2-s)}<Q$.

5.5.1 Reduction to Kloosterman sums

Lemma 40 (Dual expansion). *For $(a,q)=1$ and $\boldsymbol{u}\in(i\mathbb{R})^3$,*

$$\mathcal{E}_{\boldsymbol{u}}(a,q;W(\cdot/X))=\sum_{\boldsymbol{\varepsilon}\in\{\pm1\}^3}\sum_{\boldsymbol{h}\in\mathbb{N}^3}G(a,\boldsymbol{\varepsilon h};q)\prod_{j=1}^3\Big(\frac{h_j}{q}\Big)^{u_j/L}\frac{X}{q^3}\,k_{\boldsymbol{\varepsilon}}\Big(\frac{Xh_1h_2h_3}{q^3}\Big),\tag{33}$$

with $G(a,\boldsymbol{v};q)=\sum_{x\in(\mathbb{Z}/q)^3}e\big((ax_1x_2x_3+v_1x_1+v_2x_2+v_3x_3)/q\big)$ and

$$k_{\boldsymbol{\varepsilon}}(Y)=\frac{1}{2\pi i}\int_{(-\delta)}\widetilde{W}(w)\,Y^{w-1}\prod_{j=1}^3\gamma_{\varepsilon_j}\Big(w+\frac{u_j}{L}\Big)\,\mathrm{d}w,$$
$$\gamma_\pm(w)=\Gamma(1-w)(2\pi)^{w-1}e^{\mp\pi i(1-w)/2}.$$

For every $j\ge0$, $Y^jk_{\boldsymbol{\varepsilon}}^{(j)}(Y)\ll_j Y^{-\epsilon}$ for all $Y>0$, and $\ll_{j,B}Y^{-B}$ for $Y\ge1$; the same holds for derivatives in the parameters of W.

Proof. On $\Re w = -\delta$ every $w + u_j/L$ has negative real part, so Hurwitz's formula $\zeta(w,\theta) = \sum_{\pm} \gamma_{\pm}(w) \sum_{h\geq 1} e(\pm h\theta) h^{w-1}$ [20, (12.10)] converges absolutely; inserting it in the integral over $\Re w = -\delta$ that defines $\mathcal{E}_{\boldsymbol{u}}$ (Lemma 34) and using $\widetilde{W(\cdot/X)}(w) = X^w \widetilde{W}(w)$ gives (33). By Stirling's formula $|\gamma_{\pm}(\sigma + i\tau)| \ll (1+|\tau|)^{1/2-\sigma}$ for σ in compact sets away from the poles $w \in \{1, 2, \dots\}$, while $\widetilde{W}$ decays faster than any power of $|\Im w|$. Moving the contour to $\Re w = 1 - \epsilon$ (no poles are crossed, the poles of $\gamma_{\pm}(w + u_j/L)$ being at $\Re w \geq 1$) gives the bound $Y^{-\epsilon}$, and moving it to $\Re w = -B$ gives Y^{-B-1}. Derivatives bring down polynomial factors in w. □

Consequently the terms with $h_1h_2h_3 > NX^{\nu}$ contribute $O(X^{-B})$ for any fixed $\nu > 0$, and we may restrict to $h_1h_2h_3 \leq NX^{\nu}$.

Lemma 41 (The complete sum). *For $(a,q) = 1$ and $\boldsymbol{h} \in \mathbb{N}^3$,*

$$G(a, \boldsymbol{\varepsilon h}; q) = q \sum_{f|q} \sum_{\substack{gj=f \\ g|(h_2,h_3),\ j|(h_1,h_2h_3/g^2)}} f\, S\Big(1, -\varepsilon_1\varepsilon_2\varepsilon_3\, \overline{a}\, \frac{h_1h_2h_3}{f^2}; \frac{q}{f}\Big),$$

where $S(m,n;c)$ is the classical Kloosterman sum and $\overline{a}$ is an inverse of a modulo q/f.

Proof. Summing over x_3 gives $q\mathbf{1}[ax_1x_2 \equiv -\varepsilon_3 h_3\ (q)]$. Group x_1 by $g = (x_1, q)$, so $g \mid h_3$; write $x_1 = gx'$, $q = gq'$ with $(x', q') = 1$. The congruence becomes $x_2 \equiv -\varepsilon_3 \overline{a}\, \overline{x'} h_3/g \pmod{q'}$, and summing $e(\varepsilon_2 h_2 x_2/q)$ over the g lifts of x_2 modulo q gives $g\mathbf{1}_{g|h_2} e(-\varepsilon_2\varepsilon_3 \overline{a}\, \overline{x'} h_2h_3/(g^2q'))$. Hence $G = q\sum_{g|(q,h_2,h_3)} g\, S(\varepsilon_1 h_1, -\varepsilon_2\varepsilon_3 \overline{a} h_2h_3/g^2; q/g)$. Selberg's identity $S(m,n;c) = \sum_{j|(m,n,c)} j\, S(1, mn/j^2; c/j)$ then gives the claim, with $f = gj$. For completeness we prove the identity. If $c = c_1c_2$ with $(c_1,c_2) = 1$, then $S(m,n;c) = S(m\bar{c}_2, n\bar{c}_2; c_1)S(m\bar{c}_1, n\bar{c}_1; c_2)$, where $\bar{c}_2c_2 \equiv 1\ (c_1)$ and $\bar{c}_1c_1 \equiv 1\ (c_2)$; the right side factors in the same way, because $j \mid (m,n,c)$ splits uniquely as j_1j_2 with $j_i \mid (m,n,c_i)$, and $S(a,b;c') = S(1,ab;c')$ whenever $(a,c') = 1$. So we may take $c = p^k$. If $p \nmid m$ (or $p \nmid n$) only $j = 1$ occurs, and $S(m,n;p^k) = S(1,mn;p^k)$. Let $m = pm'$, $n = pn'$. For $k = 1$ the left side is $p-1$, and the right side is $S(1,mn;p) + p = -1 + p$. For $k \geq 2$ the left side is $pS(m',n';p^{k-1})$. On the right, the term $j = 1$ is $S(1,N;p^k)$ with $p \mid N$, which vanishes: replacing x by $x(1+p^{k-1}t)$ multiplies each term by $e(t(x - N\bar{x})/p) = e(tx/p)$, and $\sum_{t \bmod p} e(tx/p) = 0$ as $p \nmid x$. The terms $j = pj'$ give $p\sum_{j'|(m',n',p^{k-1})} j' S(1, m'n'/j'^2; p^{k-1}/j')$, which is $pS(m',n';p^{k-1})$ by induction on k. □

Write $h_1 = ju$, $h_2 = gb$, $h_3 = gc$ and $bc = jw$. Then $h_1h_2h_3 = f^2 n$ with $n = uw$, and for fixed (g, j, n) the number of (u, w, b, c) is $\alpha_j(n) = \sum_{uw=n} \tau(jw) \le \tau(j) d_3(n)$. Inserting Lemma 41 in (33), Σ becomes, up to $O(X^{-B})$,

$$\Sigma = \sum_{\varepsilon} \sum_{f \le X} \sum_{gj = f} \sum_{n \le NX^{\nu}/f^2} b_{g,j}(n) \sum_{\substack{a,q,\ (a,q)=1 \\ f \mid q,\ de = q/l}} \frac{fX}{q^3}\, V(\cdots) \times S\Big(1, \mp\varepsilon\, \overline{a} n; \frac{q}{f}\Big) k_{\varepsilon}\Big(\frac{X f^2 n}{q^3}\Big) q^{-\sum u_j/L}, \tag{34}$$

where $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3$ and $b_{g,j}(n) = \sum_{uw=n} \sum_{bc=jw} (ju)^{u_1/L} (gb)^{u_2/L} (gc)^{u_3/L}$, so that $|b_{g,j}(n)| \le \alpha_j(n)$. The factor $q^{-\sum u_j/L}$ is unimodular and smooth in q/Q. Since $X/q^3 \asymp 1/N$, the claim of Theorem 39 is that the triple sum over (a, q, d) in (34), summed over n and f, saves X^c against AQ^2, as $(X/H)/(1/N) = AQ^2$. Weil's bound gives $X^{\epsilon} AQ^{3/2} N$ for it, which suffices only when $s < \frac{2}{5}$; for larger s we need the spectral theory of $\Gamma_0(M)$.

5.5.2 The Atkin–Lehner cusp

Fix ε, $f = gj$, a dyadic range $n \asymp B$ with $1 \le B \le NX^{\nu}/f^2$, and a, d. Put $G = (ld, f)$ and $m = ld/G$. The conditions $f \mid q$ and $ld \mid q$ are equivalent to $\operatorname{lcm}(ld, f) = mf \mid q$, that is, $q = fc$ with $m \mid c$; then $e = q/(ld)$ is determined. The condition $(a, q) = 1$ is $(a, f) = 1$ together with $(a, c) = 1$, and implies $(a, m) = 1$. Put $M = am$.

Lemma 42. *Let $M = am$ with $(a, m) = 1$, choose $\beta, \delta' \in \mathbb{Z}$ with $a\delta' - m\beta = 1$, and let*

$$\sigma_{\mathfrak{a}} = W_a = a^{-1/2} \begin{pmatrix} a & \beta \\ am & a\delta' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Then $\sigma_{\mathfrak{a}}$ is a scaling matrix for the cusp $\mathfrak{a} = 1/m$ of $\Gamma_0(M)$, it normalises $\Gamma_0(M)$ (it is the Atkin–Lehner operator at the exact divisor a), the set of moduli of the pair $(\infty, \mathfrak{a})$ is $\{c\sqrt{a} :\ c \ge 1,\ m \mid c,\ (c, a) = 1\}$, and

$$S_{\infty\mathfrak{a}}(1, n; c\sqrt{a}) = S(1, n\overline{a}; c) \qquad (m \mid c,\ (c, a) = 1).$$

Proof. W_a has determinant 1, maps ∞ to $1/m$, and is the classical Atkin–Lehner matrix [21]; the stabiliser computation is standard [24, §2.3]. A product $\gamma = \begin{pmatrix} A_0 & B_0 \\ MC_0 & D_0 \end{pmatrix} W_a \in \Gamma_0(M) W_a$ has lower row $\sqrt{a}\big(m(aC_0 + D_0),\ (MC_0\beta + a\delta' D_0)/a\big)$ and upper-left entry $\sqrt{a}(A_0 + mB_0)$. Put $c =$

$m(aC_0 + D_0)$; then $m \mid c$, and $(c, a) = (D_0, a) = 1$ because $A_0 D_0 \equiv 1\ (M)$. With $x = A_0 + mB_0$ and $y = (MC_0\beta + a\delta' D_0)/a$ the Kloosterman phase is $e((x + ny)/c)$. Now $ay = MC_0\beta + a\delta' D_0 = D_0 + \beta c$ since $a\delta' = 1 + m\beta$, so $ay \equiv D_0\ (c)$, and $xD_0 = 1 + MB_0C_0 + mB_0D_0 \equiv 1\ (c)$. Hence $y \equiv \overline{a}\,\overline{x}\ (c)$, and as γ runs over the double cosets with lower-left entry $c\sqrt{a}$, x runs once over $(\mathbb{Z}/c)^*$. This gives the identity; the shift in $\sigma_{\mathfrak{a}}$ by $\begin{pmatrix} 1 & \beta/a \\ 0 & 1 \end{pmatrix}$ is exactly what removes the phase $e(n\overline{m}/a)$ that a scaling matrix without it would produce. □

Thus, for fixed (a, d, f, g, j, B), the q-sum in (34) is a sum over the moduli $c\sqrt{a}$ of the cusp pair $(\infty, \mathfrak{a})$ of $\Gamma_0(M)$, of $S_{\infty\mathfrak{a}}(1, \mp\varepsilon n; c\sqrt{a})$ against a smooth weight. By Lemma 42 the Fourier coefficients of a form u at $\mathfrak{a}$ are those of $u|W_a$ at ∞, and since a is a positive power of X the two cusps are inequivalent, so the Kuznetsov formula has no diagonal term.

5.5.3 The Kuznetsov formula

Write the weight in (34) as $(c\sqrt{a})^{-1} F(4\pi\sqrt{n}/(c\sqrt{a}))$. Then F is supported on $\xi \asymp \xi_B := K_B^{-1}$, where

$$K_B = \frac{Q\sqrt{A}}{f\sqrt{B}} \ \geq\ \frac{Q\sqrt{A}}{\sqrt{NX^{\nu}}} = KX^{-\nu/2},$$

and $\|F\|_\infty \ll X^{\epsilon} f \cdot (Q/f)\sqrt{A} = X^{\epsilon} Q\sqrt{A}$, with $\xi^i F^{(i)}(\xi) \ll_i X^{\epsilon} Q\sqrt{A}$; the factor f is the one in (34). The dependence of F on n (through $\sqrt{n}$ in the conversion) is separated by Mellin inversion in n/B on a fixed enlarged interval: $F = \int F_y(\xi)(n/B)^{iy}\,\mathrm{d}y$ with F_y of the same type and rapidly decaying in y. We apply the Kuznetsov formula for the cusp pair $(\infty, \mathfrak{a})$ of $\Gamma_0(M)$ in the form of [16, Theorem 1] (see also [24, Theorems 9.4, 9.8]): the sum is

$$\begin{aligned} &\sum_j \overline{\rho_{j\infty}(1)}\,\rho_{j\mathfrak{a}}(\pm n)\,\widetilde{F}(t_j) + \sum_{k,j} \widetilde{F}(k)\overline{\psi_{jk\infty}(1)}\psi_{jk\mathfrak{a}}(n) \\ &\qquad + \frac{1}{4\pi}\sum_{\mathfrak{c}} \int \overline{\rho_{\mathfrak{c}\infty}(1,t)}\rho_{\mathfrak{c}\mathfrak{a}}(\pm n, t)\widetilde{F}(t)\,\mathrm{d}t, \end{aligned} \tag{35}$$

over the Maass, holomorphic (only for the $+$ sign), and Eisenstein spectra, with the transforms $\widetilde{F}$ of [16]. We use the following Fourier coefficient convention throughout. Maass forms are normalised by

$\int_{\Gamma_0(M)\backslash\mathbb{H}} |u|^2\, y^{-2}\,\mathrm{d}x\,\mathrm{d}y = 1$, the Petersson inner product not divided by the volume, as in [16]. For such a Maass form u of parameter t at a cusp $\mathfrak{b}$, write

$$u(\sigma_{\mathfrak{b}} z) = \sqrt{y}\sum_{n\neq 0} \rho_{u\mathfrak{b}}^{\mathrm{raw}}(n) K_{it}(2\pi|n|y) e(nx), \qquad \rho_{u\mathfrak{b}}(n) = \frac{\rho_{u\mathfrak{b}}^{\mathrm{raw}}(n)}{\sqrt{\cosh(\pi t)}}.$$

Use the same convention for the nonconstant coefficients of Eisenstein series. In the Whittaker expansion with $W_{0,it}(4\pi|n|y) = 2\sqrt{|n|y}\, K_{it}(2\pi|n|y)$ the raw K-coefficient is $2\sqrt{|n|}$ times the Whittaker coefficient. Thus the two Maass coefficients in (35) together carry $1/\cosh(\pi t)$, including at an exceptional parameter $t = i\tau$, where $\cosh(\pi t) = \cos(\pi\tau) \geq \cos(7\pi/64) > 0.94$ for $0 < \tau \leq \theta$. In [16, Theorem 1] the Maass and Eisenstein terms are products of two raw coefficients with the weight $1/\cosh(\pi t)$, up to an absolute constant factor, which we include in $\widetilde{F}$; in (35) the weight is absorbed into the coefficients, and otherwise $\widetilde{F}$ is the transform of [16]. For a holomorphic form of weight k, $\psi_{jk\mathfrak{b}}(n)$ denotes the coefficient of $e(nz)$ at that cusp multiplied by $\sqrt{(k-1)!/(4\pi n)^{(k-1)/2}}$, the Petersson large-sieve convention of [16, (1.28)]. In [16, Theorem 1] the holomorphic term is the product of the raw coefficients at m and n times $\Gamma(k)(4\pi\sqrt{mn})^{1-k}$, up to a factor depending only on k, which we include in $\widetilde{F}(k)$ (a factor polynomial in k does not affect the bound for $\widetilde{F}(k)$ in (36) below); and $\Gamma(k)(4\pi\sqrt{n})^{1-k}$ times the raw coefficients at 1 and n is exactly $\overline{\psi_{jk\infty}(1)}\psi_{jk\mathfrak{a}}(n)$.

For $\xi_B \leq 1$ the transforms satisfy [16, Lemma 7.1], [29, Lemma 3.4]

$$\begin{gathered}\widetilde{F}(t) \ll_A X^{\epsilon} Q\sqrt{A}\,(1+|t|)^{-A}\ (t\in\mathbb{R}), \qquad \widetilde{F}(k) \ll_A X^{\epsilon} Q\sqrt{A}\,k^{-A}, \\ \widetilde{F}(i\tau) \ll X^{\epsilon} Q\sqrt{A}\,K_B^{2\tau}\ (0<\tau\leq\theta),\end{gathered} \tag{36}$$

the logarithm at $t = 0$ being absorbed in X^{ϵ}.

5.5.4 Factorisation through three L-functions

For a primitive Hecke form π (Maass or holomorphic) of conductor $M_0 \mid M$ with Hecke eigenvalues $\lambda_\pi(n)$, let $V_\pi(M)$ be the span of its old translates $u(\delta z)$, $\delta \mid M/M_0$. It is preserved by W_a, which acts unitarily on it [21].

Lemma 43 (Oldform bases). *$V_\pi(M)$ has an orthonormal basis $\{u_{\pi,\iota}\}$, $\iota \leq \tau(M/M_0)$, whose Fourier coefficients at ∞ are*

$$\rho_{u_{\pi,\iota}}(n) = \kappa_{\pi,M} \sum_{\delta \mid M/M_0} \xi_{\iota\delta}\sqrt{\delta}\,\lambda_\pi(n/\delta)\mathbf{1}_{\delta\mid n}, \qquad \xi_{\iota\delta} \ll_\epsilon M^{\epsilon},$$

where the vector with $\iota = 1$ is the newvector scaled to have L^2 norm one at level M (for the measure $y^{-2}\,\mathrm{d}x\,\mathrm{d}y$ on $\Gamma_0(M)\backslash\mathbb{H}$), and $\xi_{1\delta} = \mathbf{1}_{\delta=1}$. The factor $\sqrt{\delta}$ is the same in both conventions: $u(\delta z)$ has raw K-Bessel coefficient $\sqrt{\delta}\,\rho_u^{\mathrm{raw}}(n/\delta)$ at n, and for a holomorphic form the translate $\delta^{k/2}f(\delta z)$ has normalised coefficient $\delta^{k/2}\delta^{-(k-1)/2} = \sqrt{\delta}$ times that of f at n/δ. For Maass forms put $\kappa_{\pi,M}^{\mathrm{raw}} = \rho_{u_{\pi,1}\infty}^{\mathrm{raw}}(1)$ and $\kappa_{\pi,M} = \kappa_{\pi,M}^{\mathrm{raw}}/\sqrt{\cosh(\pi t_\pi)}$. For holomorphic forms ρ and $\kappa_{\pi,M}$ denote the Petersson-normalised coefficients and first coefficient, respectively.

Proof. This is [23, Lemma 3.15 and (3.16)] for Maass forms and [31, Theorem 8] for holomorphic forms; see also [27, §3.5]. The bound $\xi_{\iota\delta} \ll M^\epsilon$ follows from the explicit local factors there, the Kim–Sarnak bound and multiplicativity: the normalised local factors are $1 + O(p^{-1/2+\theta})$ for large p and bounded at each fixed prime uniformly in the exponent. □

Lemma 44 (Euler numerator)**.** *Let $\boldsymbol{u} \in (i\mathbb{R})^3$, $j, \delta \geq 1$, and $b_{g,j}$ as in (34). For $\Re z > 1$,*

$$\sum_{n\geq 1} b_{g,j}(\delta n)\lambda_\pi(n)n^{-z} = g^{(u_2+u_3)/L}\,\mathcal{E}_{\pi,j,\delta}(z)\prod_{i=1}^{3} L\Big(z - \frac{u_i}{L}, \pi\Big),$$

where the factor $g^{(u_2+u_3)/L}$ has modulus one, and $\mathcal{E}_{\pi,j,\delta}$ is holomorphic on $\Re z > \frac{1}{2}$ and satisfies $\mathcal{E}_{\pi,j,\delta}(z) \ll_\epsilon (j\delta M)^\epsilon$ on $\Re z \geq \frac{1}{2} + \epsilon$, uniformly in the imaginary parts. At primes $p \nmid j\delta M$ its local factor is

$$1 - (z_1z_2 + z_1z_3 + z_2z_3)p^{-2z} + \lambda_\pi(p)z_1z_2z_3p^{-3z}, \qquad z_i = p^{u_i/L}.$$

Proof. By definition $b_{g,j}(n) = g^{(u_2+u_3)/L}b_{1,j}(n)$, so it suffices to treat $g = 1$. At $p \nmid j\delta$ the local coefficients of $b_{1,j}(\delta\,\cdot)$ are, up to a unimodular constant, $b(p^v) = h_v(z_1, z_2, z_3)$, the complete homogeneous symmetric polynomial, so that $\sum_v b(p^v)Y^v = \prod_i(1 - z_iY)^{-1}$. With Satake parameters $\alpha\beta = 1$ and $\lambda_\pi(p^v) = (\alpha^{v+1} - \beta^{v+1})/(\alpha - \beta)$, the series $\sum_v h_v(\boldsymbol{z})\lambda_\pi(p^v)Y^v$ equals $(1 - e_2(\boldsymbol{z})Y^2 + \lambda_\pi(p)e_3(\boldsymbol{z})Y^3)/\prod_i(1 - \alpha z_iY)(1 - \beta z_iY)$, an identity of rational functions (also checked by computer; repeated parameters follow by continuity). The displayed numerator is $1 + O(p^{-1-2\epsilon}) + O(p^{\theta-3/2-3\epsilon})$ on $\Re z \geq \frac{1}{2}+\epsilon$, so the product over these primes converges absolutely, uniformly. At the primes dividing $j\delta M$ we keep the local series of the left side and multiply by the reciprocals of the local L-factors, which are polynomials of degree at most two in p^{-z}; the coefficient of p^{-vz} in the left local series is at most $(e+1)(b+1)^2(v+1)^2p^{\theta v}$ in modulus, where $e = v_p(j)$ and $b = v_p(\delta)$, so the local factor is bounded by a constant power of $(e+1)(b+1)$, and the product over $p \mid j\delta M$ is $\ll (j\delta M)^\epsilon$. □

Now consider the cuspidal part of (35), summed over n against $b_{g,j}(n)(n/B)^{iy}V_0(n/B)$. Since the sum over an orthonormal basis of $V_\pi(M)$ of $\overline{\rho_u(1)}\rho_{u|W_a}(n)$ does not depend on the basis, Cauchy–Schwarz within $V_\pi(M)$, unitarity of W_a and Lemma 43 bound the contribution of the class of π by

$$|\widetilde{F}(t_\pi)|\, M^\epsilon |\kappa_{\pi,M}| \Big(\sum_{\iota} \Big| \sum_{n} b_{g,j}(n) \Big(\frac{n}{B}\Big)^{iy} V_0\Big(\frac{n}{B}\Big) \rho_{u_{\pi,\iota}}(n) \Big|^2 \Big)^{1/2}.$$

For each $\delta \mid M/M_0$, Mellin inversion of V_0, a shift of the contour to $\Re z = \frac12 + \epsilon$ (there are no poles, π being cuspidal) and Lemma 44 give

$$\begin{aligned} &\sqrt{\delta} \sum_{n'} b_{g,j}(\delta n') \Big(\frac{\delta n'}{B}\Big)^{iy} V_0\Big(\frac{\delta n'}{B}\Big) \lambda_\pi(n') \\ &\qquad \ll X^\epsilon \sqrt{B} \int_{\mathbb{R}} \varpi(v) \prod_{i=1}^{3} \Big| L\Big(\tfrac12 + \epsilon + iv - \tfrac{u_i}{L}, \pi\Big)\Big| \, \mathrm{d}v, \end{aligned}$$

with ϖ rapidly decreasing, using $\sqrt{\delta}(B/\delta)^{1/2+\epsilon} \le B^{1/2+\epsilon}$. Hence the class of π contributes at most

$$X^\epsilon \sqrt{B}\, |\widetilde{F}(t_\pi)|\, |\kappa_{\pi,M}|^2 \int \varpi(v) \prod_{i=1}^{3} \Big| L\Big(\tfrac12 + \epsilon + iv - \tfrac{u_i}{L}, \pi\Big)\Big| \, \mathrm{d}v. \tag{37}$$

5.5.5 Spectral moments

Proposition 45 (Moments of L-functions). *Let $\frac12 \le \sigma \le \frac58$, $v \in \mathbb{R}$, $U \ge 1$ and $R \ge 1$.*

(i) *For each M,*

$$\sum_{\substack{\pi:\ \mathrm{cond}(\pi)\mid M \\ t_\pi \in \mathbb{R},\ |t_\pi| \le U}} \frac{|\kappa^{\mathrm{raw}}_{\pi,M}|^2}{\cosh(\pi t_\pi)} |L(\sigma + iv, \pi)|^{2k} \ll_\epsilon M^\epsilon (U + |v|)^C \qquad (k = 1, 2),$$

the sum over primitive Maass forms. The same bound holds for holomorphic forms of weight at most U with the Petersson weight $|\kappa_{\pi,M}|^2$.

(ii) *Summing over the exceptional primitive forms π, with $t_\pi = i\tau_\pi$, at all levels in a dyadic range,*

$$\begin{aligned} &\sum_{M \asymp R} \sum_{\pi \text{ exc}} |\kappa_{\pi,M}|^2 \big(R^{3\tau_\pi} |L(\sigma + iv, \pi)|^2 + R^{\tau_\pi} |L(\sigma + iv, \pi)|^4 \big) \\ &\qquad \ll_\epsilon R^{1+\epsilon} (2 + |v|)^C. \end{aligned}$$

(iii) *Consequently, for $K \geq 1$ and σ_i, v_i as above,*

$$\sum_{M \asymp R} \sum_{\pi \text{ exc}} |\kappa_{\pi,M}|^2 K^{2\tau_\pi} \prod_{i=1}^{3} |L(\sigma_i + iv_i, \pi)| \\ \ll_\epsilon R^{1+\epsilon} \max(1, K/R)^{2\theta} (2 + \max|v_i|)^C.$$

Proof. (i) Fix $M_0 \mid M$. The approximate functional equation for $L(\frac{1}{2} + iv, \pi)^k$ has two smooth sums of effective length $M_0^{k/2}(2+U+|v|)^C M^\epsilon$, with weights whose scaled derivatives are bounded by $(U+|v|)^C M^\epsilon$ (Stirling; the gamma factors are separated from their poles by the Kim–Sarnak bound) [25, Theorem 5.3]. For $k = 2$ use $L(z,\pi)^2 = \zeta^{(M_0)}(2z) \sum_n \tau(n)\lambda_\pi(n) n^{-z}$, which holds at every prime (at ramified primes $\lambda_\pi(p^j) = \lambda_\pi(p)^j$), and remove the $\zeta^{(M_0)}(2z)$ by Minkowski over $m = b^2 n$, at the cost of a logarithm. The $\iota = 1$ vectors of distinct classes are orthonormal at level M. At infinity their raw Maass coefficients are $\kappa^{\mathrm{raw}}_{\pi,M}\lambda_\pi(n)$ for $n > 0$. For a block $n \asymp Y$ the spectral large sieve [16, Theorem 2] gives

$$\sum_{\substack{j:\, t_j \in \mathbb{R} \\ |t_j| \leq U}} \frac{1}{\cosh(\pi t_j)} \Big| \sum_{n \asymp Y} a_n \rho^{\mathrm{raw}}_{j\infty}(n) \Big|^2 \ll_\epsilon (MYU)^\epsilon (U^2 + Y/M) \sum_{n \asymp Y} |a_n|^2.$$

Restricting this positive sum to the $\iota = 1$ vectors supplies precisely $|\kappa^{\mathrm{raw}}_{\pi,M}|^2/\cosh(\pi t_\pi) = |\kappa_{\pi,M}|^2$ for each class. The holomorphic large sieve gives the same bound with the Petersson-normalised coefficients. Mellin inversion in n/Y separates the smooth weights before applying either sieve. Their derivative bounds give an integrable majorant on the Mellin line, uniformly in π at fixed M_0. In both cases the squared coefficient norm of the separated block is $\ll \sum_{n \asymp Y} \tau(n)^2/n$ (up to polynomial factors in $U + |v|$ from the smooth weights). Here $Y \leq M^{1+\epsilon}(U+|v|)^C$, which gives the bound on the critical line after summing over the $\tau(M)$ divisors M_0. To the right of the line the direct sum gains $n^{1/2-\sigma}$. The dual sum loses at most $Y^{\sigma-1/2} \leq (M_0^{k/2}(2+U+|v|)^C M^\epsilon)^{\sigma-1/2}$, but it is multiplied by the factor from the functional equation of $L(\cdot,\pi)^k$ (root number, conductor and gamma factors), of modulus $\ll M_0^{k(1/2-\sigma)}(2+U+|v|)^C$. The net factor is $\ll M_0^{-k(\sigma-1/2)/2}(2+U+|v|)^C M^\epsilon \leq (2+U+|v|)^C M^\epsilon$, which gives the same bound.

(ii) Partition the conductors into dyadic ranges $M_0 \asymp R_0$. As in (i), the approximate functional equation reduces $L(\sigma+iv,\pi)^k$ to dyadic blocks of the shape $\sum_{n \asymp Y} \lambda_\pi(n) n^{-1/2-iv} V(n/Y)$ ($k = 1$, $Y \leq R_0^{1/2+\eta}(2+|v|)^C$) or

$\sum_{h\asymp Y_1,k'\asymp Y_2}\lambda_\pi(hk')(hk')^{-1/2-iv}V_1(h/Y_1)V_2(k'/Y_2)$ ($k=2$, $Y_1Y_2\le R_0^{1+\eta}(2+|v|)^C$), up to tails decaying faster than any power and a dual sum treated as in (i); the weight depending on π and M_0 is removed by Mellin inversion with uniformly rapidly decaying coefficients, after which Minkowski's inequality leaves weights V_i independent of M and π. Enlarge the sum to all $M\asymp R$ and to a full orthonormal basis of the exceptional spectrum at each level containing the $\iota=1$ vectors; all restrictions on the conductor disappear in this positive majorant. We then apply the level-averaged large sieve for exceptional eigenvalues of Watt [18], in the form of [30, Proposition 3.5]: for $Y_1\le Y_2$ and smooth V_1,V_2 whose scaled derivatives of order i are $O(Z^i)$,

$$\sum_{M\asymp R}\sum_{j\ \mathrm{exc}} X_0^{2\tau_j}\Big|\sum_{h,k'}V_1\Big(\frac{h}{Y_1}\Big)V_2\Big(\frac{k'}{Y_2}\Big)\rho_{j\infty}(hk')\Big|^2 \ll_\epsilon R^\epsilon Z^5(R+Y_1Y_2)Y_1Y_2 \qquad \Big(X_0\ll\frac{R^2}{Y_1^2Y_2}\Big). \tag{38}$$

For $k=1$ take $Y_1=1$, $Y_2=Y\le R^{1/2+\eta}(2+|v|)^C$, which permits $X_0=R^{3/2-O(\eta)}(2+|v|)^{-C}$; for $k=2$ take $Y_1\le(Y_1Y_2)^{1/2}$, which permits $X_0=R^{1/2-O(\eta)}(2+|v|)^{-C}$. After the normalisation $(Y_1Y_2)^{-1/2}$ both give $R^{1+O(\eta)}(2+|v|)^C$, and $\tau_\pi\le\theta$ restores the exact weights $R^{3\tau}$, R^τ at the cost of $R^{O(\eta)}$. Let $\eta\to0$.

(iii) Hölder's inequality with exponents $(2,4,4)$ over the joint sum, using (ii) with $k=1$ in the first factor and $k=2$ in the other two, gives the weight $R^{(\frac32+\frac14+\frac14)\tau_\pi}=R^{2\tau_\pi}$; then $K^{2\tau}\le R^{2\tau}\max(1,K/R)^{2\theta}$. □

5.5.6 The continuous spectrum

Lemma 46. *For $M=am$ with $(a,m)=1$ let $M^\flat=\prod_{p^e\|M}p^{\lceil e/2\rceil}$. With the Eisenstein coefficients normalised as in (35), for $n\neq0$,*

$$\Big|\sum_{\mathfrak{c}}\overline{\rho_{\mathfrak{c}\infty}(1,t)}\rho_{\mathfrak{ca}}(n,t)\Big|\ll_\epsilon\frac{(n,M^\flat)^{1/2}}{M^\flat}(M|n|(2+|t|))^\epsilon.$$

Proof. In [22] the Eisenstein series are expanded in the Whittaker functions $W_{0,it}(4\pi|n|y)$. Since $W_{0,it}(4\pi|n|y)=2\sqrt{|n|y}\,K_{it}(2\pi|n|y)$, a Whittaker coefficient $\rho^W(n,t)$ corresponds to the raw coefficient $2\sqrt{|n|}\,\rho^W(n,t)$, and to the coefficient $2\sqrt{|n|}\,\rho^W(n,t)/\sqrt{\cosh(\pi t)}$ in the normalisation of (35). (At level one, for example, $|\rho_{\infty\infty}(n,t)|^2=4|\sigma_{-2it}(|n|)|^2/|\zeta(1+2it)|^2$, using

$|\Gamma(\frac{1}{2}+it)|^2 = \pi/\cosh(\pi t)$.) After this conversion, [22, Lemma 1, (3.10), (4.6)] give

$$\sum_{\mathfrak{c}} |\rho_{\mathfrak{c}\infty}(n,t)|^2 \ll (n, M^\flat)(M^\flat)^{-1}(M|n|(2+|t|))^\epsilon.$$

Replacing the observation cusp ∞ by $\mathfrak{a} = W_a\infty$ permutes the Eisenstein series and acts unitarily on the continuous fibre, so the same bound holds at $\mathfrak{a}$. Cauchy–Schwarz gives the claim. □

By (36) and Lemma 46, the continuous part of (35), summed over $n \asymp B$ with $|b_{g,j}(n)| \le \alpha_j(n)$, is

$$\ll X^\epsilon Q\sqrt{A}\,\frac{1}{M^\flat}\sum_{n\asymp B}\alpha_j(n)(n,M^\flat)^{1/2} \ll X^\epsilon Q\sqrt{A}\,\frac{B}{a^\flat m^\flat},$$

using $M^\flat = a^\flat m^\flat$ (as $(a,m)=1$) and $\sum_{n\asymp B}(n,M^\flat)^{1/2} \ll B\tau(M)$. Since $d \mid mG$ and $(uv)^\flat \mid u^\flat v^\flat$, we have $1/m^\flat \le f/d^\flat$, and $\sum_{n\le Z} 1/n^\flat \ll (\log 2Z)^2$ (write $n = u^2 v$ with v squarefree, so $n^\flat = uv$). Summing over $a \asymp A$, $d \asymp D$, the dyadic $B \le NX^\nu/f^2$, and then over g, j, f with $\sum_{f\le X}\tau(f)/f \ll (\log X)^2$, the continuous contribution to the triple sum in (34) is $\ll X^{\epsilon+O(\nu)}Q\sqrt{A}\,N$. Relative to AQ^2 this is

$$X^{\epsilon+O(\nu)}\frac{N}{Q\sqrt{A}} = X^{(3s+h)/2-1+\epsilon+O(\nu)}, \tag{39}$$

which is a power saving on $\mathcal{K}$ since $s, h < \frac{1}{2}$.

5.5.7 Proof of Theorem 39

Fix $\mathfrak{e}, f = gj, B$ and $G \mid f$, and sum (37) over the levels. For fixed (l, G) the level is $M = a\,ld/G \asymp R := AlD/G$, and each M arises from at most $\tau(M)$ pairs (a,d), since $a \mid M$ determines $d = MG/(al)$. We may drop the remaining restrictions on (a,d) in this positive majorant. The three L-values in (37) have arguments independent of (a,d) at each fixed v, and all other factors are bounded by X^ϵ times integrable majorants independent of the pair. By Proposition 45(i) (with Hölder's inequality as in (iii), and rapid decay of $\widetilde{F}$ in t and k) for the regular spectrum, and Proposition 45(iii) with $K = K_B$ and (36) for the exceptional spectrum, the cuspidal contribution summed over (a,d) is

$$\ll X^{\epsilon+O(\nu)}Q\sqrt{AB}\,R\big(1+(K_B/R)^{2\theta}\big).$$

As $2\theta<1$, every power of B here is positive, so $B\le NX^{\nu}/f^2$ (which turns K_B into K up to $X^{O(\nu)}$) gives $X^{\epsilon+O(\nu)}Q\sqrt{AN}f^{-1}R(1+(K/R)^{2\theta})$. Dividing by AQ^2, with $R=AlD/G$,

$$X^{\epsilon+O(\nu)}\frac{D\sqrt{AN}}{Qf}\cdot\frac{l}{G}\Big(1+\Big(\frac{KG}{AlD}\Big)^{2\theta}\Big)\le X^{\epsilon+O(\nu)}\frac{l}{f}\cdot\frac{D\sqrt{AN}}{Q}\Big(1+\Big(\frac{K}{AD}\Big)^{2\theta}\Big),$$

using $G,l\ge1$ and $2\theta<1$. By the exponent identities (also checked by computer)

$$\frac{D\sqrt{AN}}{Q}=X^{-p},\qquad \frac{K}{AD}=X^{p},$$

this is $X^{\epsilon+O(\nu)}lf^{-1}X^{-(1-2\theta)p}$. Summing over $G\mid f$, g,j with $gj=f$, and $f\le X$ costs $\sum_f\tau(f)^2/f\ll X^{\epsilon}$; the $O(\log X)$ dyadic ranges of B and the eight sign vectors cost X^{ϵ}. With (39), the triple sum in (34) is

$$\ll AQ^2\big(l\,X^{-(1-2\theta)p}+X^{(3s+h)/2-1}\big)X^{\epsilon+O(\nu)}.$$

Choose ν,ϵ small and $c<\frac12\min_{\mathcal K}\min\big((1-2\theta)p,\,1-\frac{3s+h}{2}\big)$; since $l\le X^{c}$, this proves Theorem 39. □

Remark. The exceptional spectrum is where the level average matters. At a single level, [29, Lemma 3.8] only gives the factor $(1+K_B/M)^{2\theta}$ with $M=am$ for each pair (a,d), and the extra factor $(1+\sqrt{N/(Ald)})$ from the large sieve on the n-side; summing over d then shrinks the region to $(2k-1-x)_+<2(1-2\theta)(z-\frac12-x)$, which, added in both orientations to S_3, gives only the coefficient $1595327299882449345668448 3/384777072096051200000000=41.4610800792\ldots$. The factorisation through L-functions avoids the large sieve on the n-side altogether, and averaging over the levels $M=a\,ld/G$ gives Proposition 45(iii) with R in place of M.

5.6 Reflection and localisation

For a function g on $\mathbb R$ put $g^{\vee}(a)=\overline{g(1-a)}$, and for $T\le t\le 2T$ put

$$\ell_t=\frac{1}{L}\log\frac{t}{2\pi T}\in\Big[-\frac{2}{L},0\Big].$$

Write $\|B\|_6=(\int_T^{2T}|B(\frac12+it)|^6\,\mathrm dt)^{1/6}$. By Hölder's inequality, $\int|\Phi(t/T)B_1B_2B_3\overline{B_4}|\,\mathrm dt\le\|\Phi\|_\infty^{1/2}\|B_1\|_6\|B_2\|_6\|B_3\|_6\|B_4\|_\Phi$, and $\|\mathcal JB\|_\Phi=\|B\|_\Phi$ since $|\chi(\frac12+it)|=1$. We use Lemma 31(i) in the form $\|A_{\chi_0e^{i\xi\cdot}}\|_6\ll_{\chi_0}(TL^9)^{1/6}$ for $|\xi|\le LT/4$; the case $\xi=0$ applies to every A_h with $h\in C_c^\infty(\mathbb R)$.

Lemma 47 (Reflected factors). *Let $\chi_0 \in C_c^\infty((0,1))$ be real and $g_\xi = \chi_0 e^{i\xi\cdot}$. Uniformly for $|\xi| \le T^{1/4}$ and $T \le t \le 2T$,*

$$\chi(s)\overline{A_{g_\xi}(s)} = B_\xi(t) + O\big((1+|\xi|)^2 T^{-1/2} L\big), \quad B_\xi(t) = \sum_{k\ge 1} k^{-s}\,\overline{g_\xi}\Big(\frac{1}{L}\log\frac{t}{2\pi k}\Big), \tag{40}$$

and

$$\big\|B_\xi - A_{g_\xi^\vee}\big\|_6 \ll \frac{1+|\xi|}{L}(TL^9)^{1/6}, \qquad \big\|\chi\overline{A_{g_\xi}} - A_{g_\xi^\vee}\big\|_6 \ll \frac{(1+|\xi|)^2}{L}(TL^9)^{1/6}.$$

Proof. Put $t' = t - \xi/L$ and $s' = \frac12 + it'$. Since $A_{g_\xi}(s) = A_{\chi_0}(s')$, and the proof of Lemma 21 applies verbatim for $t' \in [T/2, 3T]$,

$$\chi(s')\overline{A_{g_\xi}(s)} = \sum_k k^{-s'}\chi_0\Big(\frac{1}{L}\log\frac{t'}{2\pi k}\Big) + O(T^{-1/2}L).$$

Write $\chi(\frac12 + i\tau) = e^{-i\vartheta(\tau)}$; by Stirling's formula $\vartheta'(\tau) = \log(\tau/2\pi) + O(\tau^{-2})$ and $\vartheta''(\tau) \ll 1/\tau$. Hence $\chi(s)/\chi(s') = \exp(-i(\xi/L)\log(t/2\pi))(1 + O((|\xi| + \xi^2)/T))$, the linear term coming from the remainder $O(\tau^{-2})$ in ϑ', and as $k^{-s'} = k^{-s}k^{i\xi/L}$,

$$\frac{\chi(s)}{\chi(s')}k^{-s'} = k^{-s}e^{-i\xi\log(t/2\pi k)/L}\big(1 + O((|\xi| + \xi^2)/T)\big).$$

Moreover $\chi_0(\log(t'/2\pi k)/L) = \chi_0(\log(t/2\pi k)/L) + O(|\xi|/(LT))$. Only $k \le 2T$ occur, and $\sum_{k\le 2T} k^{-1/2} \ll T^{1/2}$; multiplying by $\chi(s)/\chi(s')$, which has modulus 1, gives (40).

For the second claim let $G = \overline{g_\xi}$, so that $\widehat{G}(\eta) = \widehat{\chi}_0(\eta + \xi)$, and note $\frac1L\log\frac{t}{2\pi k} = 1 - \frac{\log k}{L} + \ell_t$. Choose $\omega \in C_c^\infty((0,1))$ equal to 1 on a neighbourhood of $1 - \operatorname{supp}\chi_0$; for T large, $\omega(a) = 1$ whenever $G(1 - a + \ell) \ne 0$ or $G(1-a) \ne 0$ with $|\ell| \le 2/L$. By Fourier inversion,

$$B_\xi(t) - A_{g_\xi^\vee}(s) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{G}(\eta)e^{i\eta}\big(e^{i\eta\ell_t} - 1\big)A_{\omega e^{-i\eta\cdot}}(s)\,\mathrm{d}\eta.$$

By Minkowski's inequality and $|e^{i\eta\ell_t} - 1| \le 2|\eta|/L$, the left side has $\|\cdot\|_6$ at most $\frac{1}{\pi}L^{-1}\int |\eta||\widehat{\chi}_0(\eta+\xi)|\,\|A_{\omega e^{-i\eta\cdot}}\|_6\,\mathrm{d}\eta$. For $|\eta| \le LT/4$ the last norm is $\ll (TL^9)^{1/6}$, and $\int |\eta||\widehat{\chi}_0(\eta+\xi)|\,\mathrm{d}\eta \ll 1 + |\xi|$. For $|\eta| > LT/4$ we use $\|A_{\omega e^{-i\eta\cdot}}\|_6 \ll T^{2/3}$ and $|\widehat{\chi}_0(\eta+\xi)| \ll_A |\eta|^{-A}$, valid as $|\xi| \le T^{1/4}$. The third claim follows from the first two, since the error in (40) has $\|\cdot\|_6 \ll (1+|\xi|)^2 T^{-1/3}L$. □

For a function F on $[0,1]^3$ recall $F^J(c) = F(1-c_1, 1-c_2, 1-c_3)$.

Lemma 48 (Reflected cubes). *Let $c \in C_c^\infty((0,1)^3)$ be real. Then $\|\mathcal{J}P_c - P_{c^J}\|_\Phi \ll_c L^{-1}(TL^9)^{1/2}$. Consequently, for real $\psi \in C_c^\infty(\mathcal{Y})$,*

$$\|\mathcal{J}W_\psi - P_{c_\psi^J}\|_\Phi = o\big((TL^9)^{1/2}\big), \qquad h_{c_\psi^J} = \psi^\flat \circ J.$$

Proof. Let $\chi_0 \in C_c^\infty((0,1))$ be real and equal to 1 on the three coordinate projections of $\operatorname{supp} c$, and $g_\xi = \chi_0 e^{i\xi\cdot}$. As in the proof of Lemma 31(ii), $P_c = (2\pi)^{-3}\int \widehat{c}(\xi)\prod_j A_{g_{\xi_j}}\,\mathrm{d}\xi$, so

$$\mathcal{J}P_c = (2\pi)^{-3}\int \overline{\widehat{c}(\xi)}\prod_{j=1}^3 R_{\xi_j}\,\mathrm{d}\xi, \qquad R_\xi = \chi\overline{A_{g_\xi}}.$$

Put $Q_\xi = A_{g_\xi^\vee}$. Then $(2\pi)^{-3}\int \overline{\widehat{c}(\xi)}\prod_j Q_{\xi_j}\,\mathrm{d}\xi = P_{c'}$ with

$$c'(a) = \prod_j \chi_0(1-a_j)\overline{\frac{1}{(2\pi)^3}\int \widehat{c}(\xi)e^{i\xi\cdot(1-a)}\,\mathrm{d}\xi} = \prod_j \chi_0(1-a_j)\,c(1-a) = c^J(a),$$

since c is real and $\chi_0(1-a_j) = 1$ wherever $c(1-a) \neq 0$. By Minkowski's inequality

$$\|\mathcal{J}P_c - P_{c^J}\|_\Phi \le (2\pi)^{-3}\int |\widehat{c}(\xi)|\,\Big\|\prod_j R_{\xi_j} - \prod_j Q_{\xi_j}\Big\|_\Phi\,\mathrm{d}\xi.$$

If $\max_j|\xi_j| \le T^{1/4}$, write the difference of products as $(R_1-Q_1)R_2R_3 + Q_1(R_2-Q_2)R_3 + Q_1Q_2(R_3-Q_3)$ with $R_j = R_{\xi_j}$, $Q_j = Q_{\xi_j}$. Here $\|R_j\|_6 = \|A_{g_{\xi_j}}\|_6 \ll (TL^9)^{1/6}$; $Q_j = e^{-i\xi_j}A_{\chi_0(1-\cdot)e^{i\xi_j\cdot}}$, so also $\|Q_j\|_6 \ll (TL^9)^{1/6}$; and $\|R_j - Q_j\|_6 \ll (1+|\xi|)^2L^{-1}(TL^9)^{1/6}$ by Lemma 47. Hölder's inequality bounds the norm of the difference by $\ll (1+|\xi|)^2L^{-1}(TL^9)^{1/2}$. If $\max_j|\xi_j| > T^{1/4}$ we use $|R_j|, |Q_j| \ll T^{1/2}$, and $\int_{\max|\xi_j|>T^{1/4}}|\widehat{c}| \ll_A T^{-A}$. This proves the first claim, as $\int(1+|\xi|)^2|\widehat{c}(\xi)|\,\mathrm{d}\xi < \infty$.

For the second, $c_\psi \in C_c^\infty((0,1)^3)$ is real, and

$$\begin{aligned}\|\mathcal{J}W_\psi - P_{c_\psi^J}\|_\Phi &\le \|\mathcal{J}(W_\psi - P_{c_\psi})\|_\Phi + \|\mathcal{J}P_{c_\psi} - P_{c_\psi^J}\|_\Phi \\ &= \|W_\psi - P_{c_\psi}\|_\Phi + O\big(L^{-1}(TL^9)^{1/2}\big),\end{aligned}$$

which is $o((TL^9)^{1/2})$ by Proposition 32. Finally $h_{c_\psi^J} = h_{c_\psi}\circ J = \psi^\flat \circ J$ by Lemmas 25 and 26. □

Lemma 49 (Centred functional equation)**.** *Let p be a real polynomial and $p_t(X) = p(-X - \ell_t)$. Uniformly for $T \le t \le 2T$, $\chi(s)\overline{C_p(s)} = C_{p_t}(s) + O_p(T^{-3/4})$.*

Proof. By linearity we may take $p = X^d$. Since $e^{\gamma(\frac{1}{2}+\frac{1}{L}\frac{d}{ds})}\zeta(s) = e^{\gamma/2}\zeta(s+\gamma/L)$, the polynomial $C_{X^d}(s)$ is $d!$ times the coefficient of γ^d in $e^{\gamma/2}\zeta(s+\gamma/L)$. On the critical line $\overline{\zeta(w)} = \zeta(\bar{w})$ and $\bar{s} = 1-s$, so for real γ the functional equation $\zeta(1-w) = \chi(1-w)\zeta(w)$ and $\chi(w)\chi(1-w) = 1$ give

$$\chi(s)\overline{e^{\gamma/2}\zeta(s+\gamma/L)} = e^{\gamma/2}\,\frac{\chi(s)}{\chi(s-\gamma/L)}\,\zeta(s-\gamma/L) =: G_1(\gamma).$$

Similarly $C_{p_t}(s)$ is $d!$ times the coefficient of γ^d in

$$G_2(\gamma) = e^{-\gamma(\frac{1}{2}+\frac{1}{L}\frac{d}{ds})-\gamma\ell_t}\zeta(s) = e^{-\gamma/2-\gamma\ell_t}\zeta(s-\gamma/L)$$

. Both G_1 and G_2 are holomorphic for $|\gamma| \le 1$, and as G_1 is real-analytic in real γ with Taylor coefficients $\chi\overline{C_{X^d}}/d!$, we have $\chi\overline{C_{X^d}} = G_1^{(d)}(0)$. By Stirling's formula $\chi'/\chi(\sigma+i\tau) = -\log(\tau/2\pi) + O(1/\tau)$ uniformly for $|\sigma| \le 2$, $\tau \ge 1$; integrating along the segment from $s-\gamma/L$ to s gives $\chi(s)/\chi(s-\gamma/L) = (t/2\pi)^{-\gamma/L}(1+O(1/(LT)))$, and $(t/2\pi)^{-\gamma/L} = e^{-\gamma(1+\ell_t)}$. Hence on $|\gamma| = 1$ we have $|G_1 - G_2| \ll |\zeta(s-\gamma/L)|/(LT) \ll T^{-3/4}$, by the convexity bound. Cauchy's formula on $|\gamma| = 1$ bounds $G_1^{(d)}(0) - G_2^{(d)}(0)$ by $O_d(T^{-3/4})$. □

Lemma 50 (Degree localisation)**.** *Let $\psi \in C_c^\infty(\mathcal{Y})$ be real and $\varepsilon_0 > 0$. Let $B(t) = \sum_{N \le T^C} b_t(N) N^{-1/2-it}$, where each $b_t(N)$ is smooth in $t \in [T/2, 3T]$ with $\partial_t^j b_t(N) \ll_j T^{C-j}$, and $b_t(N) = 0$ whenever $\operatorname{dist}(\log N/L, I_\psi) < \varepsilon_0$. Then $\langle B, W_\psi\rangle_\Phi \ll_A T^{-A}$.*

Proof. We have $\langle B, W_\psi\rangle_\Phi = \sum_{N,n} \overline{M_\psi(n)}(Nn)^{-1/2}\int \Phi(t/T) b_t(N)(n/N)^{it}\,\mathrm{d}t$, and $M_\psi(n) \ne 0$ only if $\log n/L \in I_\psi$, so every non-zero term has $|\log(n/N)| \ge \varepsilon_0 L$. Integrating by parts j times bounds each integral by $\ll_j T^{1+C-j}$. There are $O(T^{C+2})$ pairs and $M_\psi(n) \ll n^\epsilon$; take j large. □

Lemma 51 (Factor products)**.** *Let $f, g \in C_c^\infty((0, \frac{1}{2}))$ be real, let p be a real polynomial, and let $X = X_1X_2X_3$ with*

$$X_1 \in \{A_f,\ \chi\overline{A_f}\}, \qquad X_2 \in \{A_g,\ \chi\overline{A_g}\}, \qquad X_3 \in \{C_p,\ \chi\overline{C_p}\}.$$

Put $F(a) = F_1(a_1)F_2(a_2)F_3(a_3)$, where $F_1 = f$ if $X_1 = A_f$ and $F_1 = f^\vee$ otherwise, $F_2 = g$ or $g^\vee$ in the same way, and $F_3(a) = p(\frac{1}{2} - a)$ if $X_3 = C_p$ and $F_3(a) = p(a - \frac{1}{2})$ otherwise. Then for real $\psi \in C_c^\infty(\mathcal{Y})$

$$\langle X, W_\psi\rangle_\Phi = a_3 TL^9\Big(\int \Phi\Big)\langle h_F, \psi^\flat\rangle + o(TL^9).$$

Proof. Let Φ^+ be a fixed smooth majorant of $|\Phi|$ with the same kind of support; by Proposition 30, $\|W_\psi\|_\Phi^2 \le \langle W_\psi, W_\psi\rangle_{\Phi^+} \ll TL^9$. Throughout, an error $E(t)$ is admissible if $\int |\Phi(t/T)E\,\overline{W_\psi}|\,\mathrm{d}t = o(TL^9)$.

Step 1: explicit factors. Replace each reflected short factor $\chi\overline{A_f}$ by $B_f(t) = \sum_k k^{-s} f(\log(t/2\pi k)/L)$, using (40) with $\xi = 0$, and $\chi\overline{C_p}$ by C_{p_t}, using Lemma 49. The errors are admissible: $|A_f|, |A_g| \ll T^{1/4}$ because f, g are supported in $(0, \frac{1}{2})$, hence also $|B_f|, |B_g| \ll T^{1/4}$ by (40), $\int_T^{2T} |C_p|^2 \ll_p TL^C$, and so, for instance, an error $O(T^{-1/2}L)$ in one short factor contributes $\ll T^{-1/2}L \cdot T^{1/4}(TL^C)^{1/2}\|W_\psi\|_\Phi \ll T^{3/4}L^C$, while the error $O(T^{-3/4})$ in the third factor contributes $\ll T^{-3/4}T^{1/2} \cdot T^{1/2}\|W_\psi\|_\Phi$. Next expand C_p, respectively $C_{p_t} = \sum_{j \le \deg p}(-\ell_t)^j (j!)^{-1} C_{(p^{(j)})^-}$, by Lemma 19: up to $O(T^{-A})$ the third factor is $\sum_m \pi_t(\log m/L)\,\omega(m/T^8)m^{-s}$ with $\pi_t(a) = p(\frac{1}{2} - a)$, respectively $\pi_t(a) = p(a - \frac{1}{2} - \ell_t)$.

Step 2: localisation. Let $S_1, S_2 \subset (0,1)$ be the supports of F_1, F_2, fix ε_0 with $0 < \varepsilon_0 < \min(\frac{1}{3}, \operatorname{dist}(I_\psi, \mathbb{R} \setminus (1,2)))$, so that the sets K_3 below and $I_\psi + [-\varepsilon_0, \varepsilon_0]$ of Step 4 lie in $(-2,3)$ and $(1,2)$ respectively, and let $K_3 = I_\psi - S_1 - S_2 + [-3\varepsilon_0, 3\varepsilon_0]$. For T large a term k^{-s} of B_f has $\log k/L \in 1 - \operatorname{supp} f + [-\varepsilon_0, \varepsilon_0]$, so the first two factors only involve n_1, n_2 with $\log n_i/L \in S_i + [-\varepsilon_0, \varepsilon_0]$. Choose $\phi_3 \in C_c^\infty((-2,3))$ equal to 1 on K_3 and replace $\pi_t(\log m/L)\omega(m/T^8)$ by $\pi_t(\log m/L)\phi_3(\log m/L)$. The removed terms are products $n_1n_2m = N$ with $\operatorname{dist}(\log N/L, I_\psi) \ge \varepsilon_0$, and their coefficients are smooth in t with the derivative bounds of Lemma 50; so the change is $O(T^{-A})$.

Step 3: removing the t-dependence. After Step 2 each factor has $\|\cdot\|_6 \ll (TL^9)^{1/6}$: for A_f, A_g and $A_{\pi\phi_3}$ by Lemma 31(i), and for B_f by Lemma 47. By that lemma $\|B_f - A_{f^\vee}\|_6 \ll L^{-1}(TL^9)^{1/6}$, and in the reflected third factor the terms with $j \ge 1$ carry $|\ell_t|^j \le (2/L)^j$ and have $\|\cdot\|_6 \ll (TL^9)^{1/6}$. Telescoping and Hölder's inequality replace the product by $A_{F_1}A_{F_2}A_{F_3\phi_3}$ with an error that is $O(L^{-1}TL^9)$ after pairing with W_ψ. This product is P_c with $c(a) = F(a)\phi_3(a_3)$, and $c \in C_c^\infty(\mathbb{R}^3)$.

Step 4: evaluation. Choose $\phi_1 \in C_c^\infty((1,2))$ equal to 1 on $I_\psi + [-\varepsilon_0, \varepsilon_0]$. By Lemma 50, $\langle P_c, W_\psi\rangle_\Phi = \langle P_{c_1}, W_\psi\rangle_\Phi + O(T^{-A})$ with $c_1(a) = \phi_1(a_1 + a_2 + a_3)c(a)$, and Proposition 38 applies to c_1. Its profile is

$h_{c_1} = \phi_1(r) h_{F\phi_3(c_3)}$, because $\kappa(x,y,z;c)$ vanishes unless $c_1+c_2+c_3 = r$. On $\{r \in I_\psi\}$, which contains the support of $\psi^\flat$, the factor $\phi_1(r)$ is 1, and every c with $\kappa(x,y,z;c)F(c) \neq 0$ has $c_i \in S_i$ ($i = 1,2$) and $c_3 = r - c_1 - c_2 \in K_3$, where $\phi_3 = 1$. Hence $\langle h_{c_1}, \psi^\flat\rangle = \langle h_F, \psi^\flat\rangle$. □

5.7 Proof of Theorem 23

Fix $\Phi \in \mathcal{W}$. All tests in $\mathcal{T}_3^\infty$ have the form ΠK, with K one of

$$D_\phi, \qquad U = \chi C_p A_f \overline{A_g} = A_f \cdot \chi\overline{A_g} \cdot C_p, \qquad V = C_p A_f A_g,$$

the last with $f \in C_c^\infty((0,a_0))$, $g \in C_c^\infty((0,b_0))$ and $a_0 + b_0 < \frac12$; for U, $a_0 < \frac14$ and $b_0 < \frac25$ since $3a_0 + 5b_0 < 2$. Recall $\Pi K = \frac12(K + \mathcal{J}K)$, $h_{\Pi K} = \frac12(h_{F_K} + h_{F_K}\circ J)$, and that $\mathcal{J}$ is an involution with $\langle \mathcal{J}H, \mathcal{J}K\rangle_\Phi = \overline{\langle H, K\rangle_\Phi}$ for real Φ; in particular $\mathcal{J}\Pi K = \Pi K$ and $\mathcal{J}\zeta^3 = \zeta^3$.

The span. The profiles of V-tests are uniform limits of profiles of V'-tests: if f, g are the polynomial data of a V-test on $[0,\eta]$, $\eta < \frac14$, choose $f_n, g_n \in C_c^\infty((0,\eta))$ converging uniformly to f, g, which is possible because f, g vanish at 0 and η; with $a_0 = b_0 = \eta$ these are V'-data, and $|h_F| \le \|F\|_\infty v$ (Lemma 6(ii)). Hence $\mathcal{H}(\mathcal{T}_3^\infty) = \mathcal{H}(\mathcal{T}_3)$, and $v\mathbf{1}_{S_3} \in \mathcal{H}(\mathcal{T}_4)$ by Theorem 16. Let $\mathcal{A}^\flat = \{(x, k-x, r-k) : (r,k,x) \in \mathcal{Y}\}$, which by the description of $\mathcal{Y}$ is $\Omega \cap \mathcal{D}_+ \cap \mathcal{Z} \cap \{1 < r < 2,\ k > \frac12\}$. The functions $\psi^\flat$, $\psi \in C_c^\infty(\mathcal{Y})$ real, are dense in $L^2(\mathcal{A}^\flat)$, and the $\psi^\flat \circ J$ are dense in $L^2(J\mathcal{A}^\flat)$; so $\mathcal{H}(\mathcal{T}_4) \supseteq L^2(\mathcal{A}^\flat \cup J\mathcal{A}^\flat)$. We claim that $\mathcal{Z} \setminus S_3 \subseteq \mathcal{A}^\flat \cup J\mathcal{A}^\flat$ up to a null set. Let $(x,y,z) \in \mathcal{Z} \setminus S_3$ lie off the null set where one of the defining inequalities is an equality. As it is not in $\mathcal{A}$, $1 < r < 2$; as it is not in $\mathcal{D}_{\mathrm{acc}}$, $x + y > \frac12$ and $y + z < \frac32$. If $x + y < z$ it lies in $\mathcal{A}^\flat$. Otherwise it lies in $J\mathcal{D}_+$, and its image under J lies in $\mathcal{D}_+ \cap \mathcal{Z}$, has trace $3 - r \in (1,2)$ and has $(1-z) + (1-y) = 2 - y - z > \frac12$, so lies in $\mathcal{A}^\flat$. Hence $v\mathbf{1}_{\mathcal{Z}\setminus S_3} \in \mathcal{H}(\mathcal{T}_4)$, and so $v\mathbf{1}_{S_3 \cup \mathcal{Z}} = v\mathbf{1}_{S_3} + v\mathbf{1}_{\mathcal{Z}\setminus S_3} \in \mathcal{H}(\mathcal{T}_4)$.

Faithfulness. Pairs within $\mathcal{T}_3^\infty$ are covered by Theorem 16. Let $\psi, \eta \in C_c^\infty(\mathcal{Y})$ be real.

(a) *Numerators.* By Lemma 19, $\zeta(s)^3 = \sum_m d_3(m)\omega(m/T^8)m^{-s} + O(T^{-A})$. With ϕ_1 as in Step 4 of the proof of Lemma 51, Lemma 50 removes the terms with $\phi_1(\log m/L) \neq 1$, so $\mathrm{N}_T(W_\psi) = \langle P_{c_0}, W_\psi\rangle_\Phi + O(T^{-A})$, where $c_0(a) = \phi_1(a_1+a_2+a_3)\omega_0(a)$ and $\omega_0 \in C_c^\infty(\mathbb{R}^3)$ is 1 on $[0,2]^3$. By Proposition 38, $\mathrm{N}_T(W_\psi)/\mathcal{N}_T \to \langle h_{c_0}, \psi^\flat\rangle = \langle \phi_1(r)v, \psi^\flat\rangle = \langle v, \psi^\flat\rangle$, using $h_1 = v$. Next, $\mathrm{N}_T(\mathcal{J}W_\psi) = \langle \mathcal{J}\zeta^3, \mathcal{J}W_\psi\rangle_\Phi = \overline{\mathrm{N}_T(W_\psi)}$, whose normalised limit is $\langle v, \psi^\flat\rangle = \langle v, \psi^\flat \circ J\rangle$ because $v \circ J = v$ and J preserves measure.

(b) *Marked Gram entries.* $\mathrm{G}_T(W_\psi, W_\eta)/\mathcal{N}_T \to \langle \psi^\flat, \eta^\flat\rangle$ by Proposition 30, and $\mathrm{G}_T(\mathcal{J}W_\psi, \mathcal{J}W_\eta) = \overline{\mathrm{G}_T(W_\psi, W_\eta)}$ has the limit $\langle \psi^\flat, \eta^\flat\rangle = \langle \psi^\flat \circ J, \eta^\flat \circ J\rangle$.

For the mixed orientation, Lemma 48 (which rests on Proposition 32) gives $\|\mathcal{J}W_\eta - P_{c_\eta^J}\|_\Phi = o((TL^9)^{1/2})$, so by Cauchy–Schwarz and the bound $\|W_\psi\|_\Phi \ll (TL^9)^{1/2}$, $\mathrm{G}_T(\mathcal{J}W_\eta, W_\psi) = \langle P_{c_\eta^J}, W_\psi\rangle_\Phi + o(TL^9)$. Every $a \in \operatorname{supp} c_\eta^J$ has trace in $3 - I_\eta \subset (1,2)$, so Proposition 38 gives the limit $\langle h_{c_\eta^J}, \psi^\flat\rangle = \langle \eta^\flat \circ J, \psi^\flat\rangle$; and $\mathrm{G}_T(W_\psi, \mathcal{J}W_\eta)$ is its complex conjugate.

(c) *Cross-pairings with $\mathcal{T}_3^\infty$.* Since $\mathrm{G}_T(\Pi K, \mathcal{J}W_\psi) = \overline{\mathrm{G}_T(\mathcal{J}\Pi K, W_\psi)} = \overline{\mathrm{G}_T(\Pi K, W_\psi)}$ and $h_{\Pi K}$ is J-symmetric, it suffices to show

$$\frac{\mathrm{G}_T(K, W_\psi)}{\mathcal{N}_T} \to \langle h_{F_K}, \psi^\flat\rangle, \qquad \frac{\mathrm{G}_T(\mathcal{J}K, W_\psi)}{\mathcal{N}_T} \to \langle h_{F_K} \circ J, \psi^\flat\rangle$$

for each K; then $\mathrm{G}_T(\Pi K, W_\psi)/\mathcal{N}_T \to \langle h_{\Pi K}, \psi^\flat\rangle$.

For $K = U$ and $K = V$ these are Lemma 51 with

$$U = A_f \cdot \chi\overline{A_g} \cdot C_p, \qquad \mathcal{J}U = \chi\overline{A_f} \cdot A_g \cdot \chi\overline{C_p},$$
$$V = A_f \cdot A_g \cdot C_p, \qquad \mathcal{J}V = \chi\overline{A_f} \cdot \chi\overline{A_g} \cdot \chi\overline{C_p},$$

whose weights F in Lemma 51 are F_U, F_U^J, F_V and F_V^J respectively; and $h_{F^J} = h_F \circ J$ by Lemma 26.

For $K = D_\phi$ with $\operatorname{supp}\phi \subset (0,\theta)$, $\theta < 1$, the profiles $h_{F_\phi} = \phi(r)v$ and $h_{F_\phi} \circ J = \phi(3-r)v$ vanish on $1 < r < 2$, so both limits must be 0. All coefficients of D_ϕ have $\log N/L < \theta$, at a fixed distance from I_ψ, so $\mathrm{G}_T(D_\phi, W_\psi) = O(T^{-A})$ by Lemma 50. For the reflected entry, $\mathrm{G}_T(\mathcal{J}D_\phi, W_\psi) = \int \Phi(t/T)\chi(s)^3\overline{D_\phi(s)W_\psi(s)}\,\mathrm{d}t$, and $D_\phi W_\psi = \sum_N e(N)N^{-s}$ with $e(N) \ll N^\epsilon$ supported on $N \le T^{\theta+\rho}$, where $\rho = \max I_\psi < 2$. The phase of $\chi(s)^3 N^{it}$ is $-3\vartheta(t) + t\log N$, whose derivative is at most $-(3-\theta-\rho)L + O(1)$, while its higher derivatives are $O(T^{1-j})$. Repeated integration by parts gives $O_A(T^{-A})$ for each N, hence in total.

All limits hold for every $\Phi \in \mathcal{W}$, so $\mathcal{T}_4$ is faithful.

The measure. Up to null sets $\Omega \setminus S_3 = \mathcal{B} \setminus \mathcal{A}$ (see the proof of Theorem 16), so

$$\Omega \setminus (S_3 \cup \mathcal{Z}) = \big\{1 < r < 2,\ x + y > \tfrac12,\ y + z < \tfrac32,\ z - x < \tfrac12,\\ 3\min(x,\varrho) + 5\max(x,\varrho) > 2\big\}.$$

The last condition is implied by the others, since $3\min(x,\varrho) + 5\max(x,\varrho) \ge 4(x+\varrho) = 4(1-(z-x)) > 2$. If $r < \frac32$ then $y + z = r - x < \frac32$ automatically, and if $r > \frac32$ then $x + y = r - z > \frac12$ automatically. Hence the set is the disjoint union, up to a null set, of N_0 and $JN_0 = \{\frac32 < r < 2,\ y + z < \frac32,\ z - x < \frac12\}$, and $\mu(\Omega \setminus (S_3 \cup \mathcal{Z})) = 2\mu(N_0)$ by Lemma 4. Finally $\mu(N_0) = \frac{209}{1024}$: N_0 is a convex polytope, and integrating the polynomial $9!\,v^2$ exactly over a triangulation of it gives this value (numerical quadrature confirms it). □

6 The balanced region

In this section we add profiles supported on a region of shapes of trace just above 1 whose two shorter coordinates have sum just above $\frac{1}{2}$. Put $|a| = a_1 + a_2 + a_3$ for $a \in \mathbb{R}^3$, and let

$$\mathcal{R} = \{(x,y,z) \in \Omega : \ 1 < r < \tfrac{15}{14},\ 21k + 7r < 18\},$$
$$\mathcal{R}^\sharp = \{a \in (0,1)^3 : \ 1 < |a| < \tfrac{15}{14},\ 21(a_1 + a_2) + 7|a| < 18\}.$$

Thus $\mathcal{R} = \mathcal{R}^\sharp \cap \Omega$, and $a_1 + a_2 < \frac{18-7}{21} = \frac{11}{21}$ on $\mathcal{R}^\sharp$. The sets $\mathcal{R}$ and $J\mathcal{R}$ are disjoint, since the trace is below $\frac{15}{14}$ on the first and above $\frac{27}{14}$ on the second. Write ∂_j for the derivative in the jth coordinate and

$$\mathsf{V} = (\partial_2 - \partial_1)(\partial_3 - \partial_1)(\partial_3 - \partial_2).$$

For real $\psi \in C_c^\infty(\mathcal{R})$ the function $\mathsf{V}\psi$ is real and smooth with support in $\operatorname{supp}\psi \subset \mathcal{R}^\sharp$, and we put

$$B_\psi = P_{\mathsf{V}\psi}, \qquad I_\psi = \{x + y + z : \ (x,y,z) \in \operatorname{supp}\psi\} \subset (1, \tfrac{15}{14}),$$

in the notation $P_c(s) = \sum_{n_1,n_2,n_3} c(\ell(n))(n_1 n_2 n_3)^{-s}$ of Section 5. So B_ψ is a product-type Dirichlet polynomial in three factors whose lengths $T^{a_1} \le T^{a_2} \le T^{a_3}$ are distributed according to $\mathsf{V}\psi$.

Theorem 52 (Balanced row)**.** *Let*

$$\mathcal{T}_{\mathcal{R}} = \big\{(B_\psi, \psi),\ (\mathcal{J}B_\psi, \psi \circ J) : \ \psi \in C_c^\infty(\mathcal{R}) \text{ real}\big\}, \qquad \mathcal{T}_5 = \mathcal{T}_4 \cup \mathcal{T}_{\mathcal{R}}.$$

Then $\mathcal{T}_5$ is faithful, $v\mathbf{1}_{S_3 \cup \mathcal{Z} \cup \mathcal{R} \cup J\mathcal{R}} \in \mathcal{H}(\mathcal{T}_5)$, and

$$\begin{aligned}\mu(S_3 \cup \mathcal{Z} \cup \mathcal{R} \cup J\mathcal{R}) &= 42 - 2\mu(N_0) + 2\mu(N_0 \cap \mathcal{R}) \\ &= C_0 := \frac{2578241177999}{61983140352} = 41.5958462794\ldots.\end{aligned}$$

The profile of B_ψ is ψ by a Jacobi–Trudi identity (Lemma 53). The analytic input is Proposition 58: the pairing of any P_c with c supported in $\mathcal{R}^\sharp$ against an arbitrary smooth cubic source is given by the Schur inner product. To prove it we group the two shorter factors of P_c into a modulus $q \asymp T^{a_1+a_2}$, write the time integral as a sum over $h = n - qm$, and replace each progression $n \equiv h \pmod q$ by the mean over its gcd class. This uses an estimate for d_3 in progressions with smooth weights (Proposition 55) deduced from Heath-Brown's work [15], and it is here that the conditions defining $\mathcal{R}$ enter (Lemma 56). The gcd-class means are then evaluated by a contour shift and Poisson summation in h. The result is a four-shift Euler product, whose limit we identify with the Schur inner product by a four-variable Haar identity (Lemmas 65 and 66).

6.1 Synthesis

Lemma 53 (Synthesis). *For $\psi \in C_c^\infty(\Omega)$ we have $h_{\mathsf{V}\psi} = \psi$ on Ω.*

Proof. Step 2 of the proof of Lemma 14 uses only that $\psi \in C_c^\infty(\Omega)$; the box Q there plays no role in that step. It shows that $h_\Psi = \psi$, where

$$\Psi = \frac{1}{6}\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\big((\iota - \sigma)\cdot\nabla\big)^3 \hat{\psi}, \qquad \hat{\psi}(m_1,m_2,m_3) = \psi(m_3,m_2,m_1).$$

For $t \in \mathbb{R}^3$, expanding a determinant,

$$\sum_{\sigma\in S_3} \operatorname{sgn}(\sigma) e^{(\iota-\sigma)\cdot t} = e^{\iota\cdot t}\det\left(e^{-jt_i}\right)_{i,j=1}^3 = e^{\iota\cdot t - t_1 - t_2 - t_3}\prod_{1\le i<l\le 3}\left(e^{-t_l} - e^{-t_i}\right).$$

The left side has no terms of degree below three (as noted in Step 2), and the right side has cubic part $\prod_{i<l}(t_i - t_l)$. Comparing cubic parts gives $\frac{1}{6}\sum_\sigma \operatorname{sgn}(\sigma)((\iota-\sigma)\cdot t)^3 = \prod_{i<l}(t_i - t_l)$, so $\Psi = (\partial_{m_1} - \partial_{m_2})(\partial_{m_1} - \partial_{m_3})(\partial_{m_2} - \partial_{m_3})\hat{\psi}$. Since $\partial_{m_1}, \partial_{m_2}, \partial_{m_3}$ act on $\hat{\psi}$ as $\partial_3, \partial_2, \partial_1$ act on ψ, this says $\Psi(c_1,c_2,c_3) = (\mathsf{V}\psi)(c_3,c_2,c_1)$. Finally $h_{F\circ\pi} = h_F$ for every permutation π of the coordinates (Step 3 of the proof of Lemma 14), so $h_{\mathsf{V}\psi} = h_\Psi = \psi$. $\square$

The identity has also been checked by numerical quadrature.

6.2 The divisor function in progressions

For $q \ge 1$, a unit $a \bmod q$ and a box $\mathcal{B} = I_1 \times I_2 \times I_3$ of integer intervals put

$$\Delta_q(\mathcal{B}; a) = \#\{n \in \mathcal{B} :\ n_1 n_2 n_3 \equiv a\ (q)\} - \frac{1}{\varphi(q)}\#\{n \in \mathcal{B} :\ (n_1 n_2 n_3, q) = 1\}.$$

For $U, V, W \ge 1$ let $\Delta_q^*(U,V,W)$ be the supremum of $|\Delta_q(\mathcal{B};a)|$ over units a and boxes with $I_1 \subseteq [1,4U]$, $I_2 \subseteq [1,4V]$, $I_3 \subseteq [1,4W]$. It is nondecreasing in each of U, V, W.

Lemma 54 (Heath-Brown). *Let $\epsilon > 0$, $Y \ge 2$ and $U, V, W \ge 1$ with $UVW \le Y$. For $1 \le q \le Y$,*

(i) $\Delta_q^*(U,V,W) \ll_\epsilon Y^\epsilon\big(q^{5/6} + q^{1/4}(UV)^{1/2} + q^{1/2}W^{1/2}\big)$;

(ii) $\Delta_q^*(U,V,W) \ll_\epsilon Y^\epsilon q^{1/2}U$;

and for $1 \le Q \le Y$,

(iii) $\sum_{Q<q\le 2Q}\Delta_q^*(U,V,W) \ll_\epsilon Y^\epsilon Q^{3/4}U^{3/4}V^{1/2}(W+U)^{1/4}(V^2+Q)^{1/4}$.

Source. In [15, Lemmas 5 and 6] the number of n in a box with $n_1n_2n_3$ in a given unit class is compared with an approximant that does not depend on the class, with errors bounded by the right sides of (i) and (ii). The proofs use, for an interval of length I and a frequency $0 < |j| \le q/2$, only the bound $\min(I, q/|j|)$ for the corresponding geometric sum, so they apply to every box in the stated range, and the bounds increase with the side lengths. If $|S(a) - M| \le E$ for every unit a, then also $|S(a) - \varphi(q)^{-1}\sum_b S(b)| \le 2E$; this replaces the approximant by the class mean. For (iii), the proof of [15, Lemma 8], through [15, (6.2)–(6.5)], bounds the discrepancy for a fixed modulus by quantities that depend on the box only through the factor $H(I_2, q)^{1/4}$, where $H(I, q)$ counts quadruples of units in I with equal sums of reciprocal pairs modulo q. Since $H(I_2, q) \le H([1, 4V], q)$, the supremum over classes and boxes may be taken inside the sum over q, and [15, (6.14)] bounds $\sum_{Q<q\le 2Q} H([1, 4V], q)$ by $Y^{\epsilon}(V^4 + QV^2)$. Hölder's inequality as in [15, Lemma 8] gives (iii). □

Proposition 55 (Smooth discrepancy)**.** *Let $Q \ge 1$, $U, V, W > 0$ and $h \in \mathbb{Z}\setminus\{0\}$. For each $Q < q \le 2Q$ let $w_q \in C^\infty(\mathbb{R}^3)$ be supported in $[U, 4U] \times [V, 4V] \times [W, 4W]$, with $|(y_1\partial_1)(y_2\partial_2)(y_3\partial_3)w_q(y)| \le M$ for all y. Put $g = (h, q)$ and*

$$D_q(h; w) = \sum_{\substack{n\in\mathbb{N}^3 \\ n_1n_2n_3\equiv h\ (q)}} w(n) - \frac{1}{\varphi(q/g)} \sum_{\substack{n\in\mathbb{N}^3 \\ (n_1n_2n_3, q)=g}} w(n).$$

Write $U^+ = \max(U, 1)$ and similarly V^+, W^+, and let $Y = 2 + QU^+V^+W^+|h|$. Then

$$\sum_{Q<q\le 2Q} |D_q(h; w_q)| \ll_\epsilon MY^\epsilon Q \min_\pi \min\big(E_1, E_2, E_3\big)\big(\pi(U^+, V^+, W^+)\big),$$

where π runs over the permutations of the three arguments and

$$E_1(U, V, W) = Q^{5/6} + Q^{1/4}(UV)^{1/2} + Q^{1/2}W^{1/2}, \qquad E_2(U, V, W) = Q^{1/2}U,$$
$$E_3(U, V, W) = U^{3/4}V^{1/2}(W + U)^{1/4}\big(1 + V^{1/2}Q^{-1/4}\big).$$

Proof. (a) Unit classes. Let $q_1 \ge 1$, let a be a unit modulo q_1, and let w be smooth with support in $[U', 4U'] \times [V', 4V'] \times [W', 4W']$ and $|(y_1\partial_1)(y_2\partial_2)(y_3\partial_3)w| \le M$. Put

$$D^\circ_{q_1}(a; w) = \sum_{n_1n_2n_3\equiv a\ (q_1)} w(n) - \frac{1}{\varphi(q_1)} \sum_{(n_1n_2n_3, q_1)=1} w(n).$$

Since w vanishes with all its derivatives on the lower faces of the box, $w(y) = \int_{U'}^{y_1}\int_{V'}^{y_2}\int_{W'}^{y_3} \partial_1\partial_2\partial_3 w(t)\,\mathrm{d}t$ for y in the box. Inserting this and interchanging the finite sum with the integral,

$$D^{\circ}_{q_1}(a;w) = \int_{[U',4U']\times[V',4V']\times[W',4W']} \partial_1\partial_2\partial_3 w(t)\,\Delta_{q_1}(\mathcal{B}_t;a)\,\mathrm{d}t,$$

where $\mathcal{B}_t$ is the box of integer points n with $t_j \le n_j \le 4U'_j$, writing $(U'_1,U'_2,U'_3) = (U',V',W')$. Here $\mathcal{B}_t \subseteq [1,4U'^{+}]\times[1,4V'^{+}]\times[1,4W'^{+}]$ because $n_j \ge 1$, and $|\partial_1\partial_2\partial_3 w(t)| \le M/(t_1t_2t_3) \le M/(U'V'W')$. The box has volume $27U'V'W'$, so

$$|D^{\circ}_{q_1}(a;w)| \le 27M\,\Delta^{*}_{q_1}(U'^{+},V'^{+},W'^{+}). \tag{41}$$

(b) Reduction to units. Fix q with $Q < q \le 2Q$ and put $g = (h,q)$, $q = gq_1$, $h = gh_1$, so that $(h_1,q_1) = 1$. If $n_1n_2n_3 \equiv h \pmod q$ or $(n_1n_2n_3,q) = g$, then $g \mid n_1n_2n_3$. For such n put

$$e_1 = (n_1,g), \qquad e_2 = (n_2, g/e_1), \qquad e_3 = g/(e_1e_2).$$

For each prime one of n_1/e_1 and g/e_1 is not divisible by it, so $(n_1/e_1, g/e_1) = 1$; as g/e_1 divides $(n_1/e_1)n_2n_3$, it divides n_2n_3. In the same way $(n_2/e_2, e_3) = 1$ and $e_3 \mid n_3$. The map $n \mapsto (e,n')$, $n'_j = n_j/e_j$, is a bijection from the n with $g \mid n_1n_2n_3$ onto the pairs with $e_1e_2e_3 = g$, $(n'_1, e_2e_3) = 1$ and $(n'_2,e_3) = 1$; the inverse is $n_j = e_jn'_j$, and indeed $(e_1n'_1, g) = e_1(n'_1, e_2e_3) = e_1$ and $(e_2n'_2, e_2e_3) = e_2$. In these variables $n_1n_2n_3 \equiv h \pmod q$ becomes $n'_1n'_2n'_3 \equiv h_1 \pmod{q_1}$, and $(n_1n_2n_3,q) = g$ becomes $(n'_1n'_2n'_3, q_1) = 1$; the $\varphi(q/g) = \varphi(q_1)$ residue classes modulo q with gcd g correspond to the units modulo q_1. Remove the conditions $(n'_1,e_2e_3) = 1$ and $(n'_2,e_3) = 1$ by Möbius inversion over $t_1 \mid e_2e_3$ and $t_2 \mid e_3$, writing $n'_1 = t_1n''_1$ and $n'_2 = t_2n''_2$. If $(t_1t_2,q_1) > 1$ both terms of the discrepancy vanish, because h_1 is a unit. Otherwise $t_1t_2n''_1n''_2n'_3 \equiv h_1$ is the unit class $n''_1n''_2n'_3 \equiv h_1\overline{t_1t_2}$, and the coprimality condition is unchanged. Hence

$$D_q(h;w_q) = \sum_{e_1e_2e_3=g}\ \sum_{\substack{t_1\mid e_2e_3,\ t_2\mid e_3\\ (t_1t_2,q_1)=1}} \mu(t_1)\mu(t_2)\,D^{\circ}_{q_1}\big(h_1\overline{t_1t_2}; w_q^{e,t}\big),$$

with $w_q^{e,t}(y) = w_q(e_1t_1y_1, e_2t_2y_2, e_3y_3)$. This weight is supported in a box of the form in (a), with $U' = U/(e_1t_1)$, $V' = V/(e_2t_2)$, $W' = W/e_3$, and satisfies the same derivative bound because each $y_j\partial_j$ is invariant under dilation. Since $U'^{+} \le U^{+}$, $V'^{+} \le V^{+}$, $W'^{+} \le W^{+}$, (41) and monotonicity of Δ^{*} give $|D_q(h;w_q)| \le 27M\,\tau_3(g)\tau(g)^2\,\Delta^{*}_{q_1}(U^{+},V^{+},W^{+})$.

(c) Summation. Each q determines $g \mid h$, and $\tau_3(g)\tau(g)^2 \ll Y^\epsilon$. Thus

$$\sum_{Q<q\le 2Q} |D_q(h;w_q)| \ll MY^\epsilon \sum_{g\mid h} \sum_{Q/g<q_1\le 2Q/g} \Delta^*_{q_1}(U^+,V^+,W^+).$$

The term $q_1 = 1$ vanishes, since there is only one class modulo 1, so we may assume $2Q/g \ge 2$. By Lemma 54(i), (ii), summed over q_1, and (iii), the inner sum is $\ll Y^\epsilon(Q/g)E_i$ for $i = 1,2,3$, evaluated at Q/g in place of Q and at any permutation of (U^+,V^+,W^+). Each of QE_1, QE_2, QE_3 is nondecreasing in Q (for the last, $QE_3 \asymp Q^{3/4}U^{3/4}V^{1/2}(W+U)^{1/4}(V^2+Q)^{1/4}$), so we may replace Q/g by Q. Summing over the $\tau(h) \ll Y^\epsilon$ divisors g proves the proposition. □

Remark. For $h = 0$ the discrepancy vanishes identically: the class $0 \bmod q$ is the only class with gcd q, and $\varphi(1) = 1$.

Lemma 56 (Exponents). *For $k \ge 0$ and $a \in [0,\infty)^3$ put*

$$\theta_1(k;a) = \max\left(\tfrac{11k}{6},\ \tfrac{5k}{4} + \tfrac{a_1+a_2}{2},\ \tfrac{3k}{2} + \tfrac{a_3}{2}\right), \qquad \theta_2(k;a) = \tfrac{3k}{2} + a_1,$$
$$\theta_3(k;a) = \max\left(k + \tfrac{3a_1}{4} + \tfrac{a_2}{2} + \tfrac{\max(a_1,a_3)}{4},\ \tfrac{3k}{4} + \tfrac{3a_1}{4} + a_2 + \tfrac{\max(a_1,a_3)}{4}\right),$$

and $\theta(k;a) = \min_\pi \min_i \theta_i(k;\pi a)$ over permutations π of the coordinates. Let

$$\mathcal{O} = \{(k,r) :\ k \ge 0,\ 1 < r < \tfrac{15}{14},\ 21k + 7r < 18\},$$

and let $K \subset \mathcal{O}$ be compact. There are $\eta, \delta_0 > 0$ such that $\theta(\max(k,\frac12);a) \le 1-\eta$ whenever $(k,r) \in K$, $a \in [0,\infty)^3$ and $|a_1+a_2+a_3-r| \le \delta_0$.

For $Q = T^k$ and $(U,V,W) = (T^{a_1},T^{a_2},T^{a_3})$ we have $QE_i(U,V,W) \le 3T^{\theta_i(k;a)}$ (for $i = 1$ each of the three summands is at most T^{θ_1}, and for $i = 3$ the factor is at most $2\cdot 2^{1/4}$), and each θ_i is nondecreasing in k.

Proof. If $k < \frac12$ then $(\frac12, r) \in \mathcal{O}$, because $\frac{21}{2} + 7r < 18$ is $r < \frac{15}{14}$. So it suffices to prove $\theta(k;a) < 1$ when $(k,r) \in \mathcal{O}$, $k \ge \frac12$, $a \ge 0$ and $|a| = r$. The lemma then follows because $(k,r,a) \mapsto \theta(\max(k,\frac12);a)$ is continuous, the set of (k,r,a) with $(k,r) \in K$, $a \ge 0$, $|a| = r$ is compact, and θ is uniformly continuous on a compact neighbourhood of it.

So let $k \ge \frac12$, $1 < r < \frac{15}{14}$, $21k + 7r < 18$ and $|a| = r$. Put $P = r + \frac52 k - 2$ and $U_* = 2 - 3k$. Substituting $k < (18-7r)/21$ and using $1 < r < \frac{15}{14}$ one finds

$$k < \tfrac{11}{21}, \qquad P < \tfrac{r}{3}, \qquad P < U_*, \qquad r + \tfrac{15}{2}k < 5, \qquad 19k + 6r < 16;$$

for instance $\frac{2}{3}r+\frac{5}{2}k<\frac{90-7r}{42}<2$ gives $P<\frac{r}{3}$, and $r+\frac{11}{2}k<\frac{198-35r}{42}<4$ gives $P<U_*$.

Case 1: some coordinate w of a lies in (P,U_*). Put it last. Then $\theta_1=\max\big(\frac{11k}{6},\frac{5k}{4}+\frac{r-w}{2},\frac{3k}{2}+\frac{w}{2}\big)$. Here $\frac{11k}{6}<\frac{121}{126}<1$, $\frac{5k}{4}+\frac{r-w}{2}<\frac{5k}{4}+\frac{r-P}{2}=1$ and $\frac{3k}{2}+\frac{w}{2}<\frac{3k}{2}+\frac{U_*}{2}=1$.

Case 2: every coordinate is at most P or at least U_*. Order them as $\alpha\le\beta\le\gamma$. Since $\gamma\ge\frac{r}{3}>P$, we have $\gamma\ge U_*$. If $\beta\ge U_*$, then $\alpha=r-\beta-\gamma\le r-2U_*$, and θ_2 with α in the first slot is $\frac{3k}{2}+\alpha\le r+\frac{15}{2}k-4<1$. Otherwise $\alpha\le\beta\le P$, and we use θ_3 with the slots (β,α,γ). As $\gamma\ge\beta$, its first term is

$$k+\tfrac{3\beta}{4}+\tfrac{\alpha}{2}+\tfrac{\gamma}{4}=k+\tfrac{r}{4}+\tfrac{\alpha+\beta}{4}+\tfrac{\beta}{4}\le k+\tfrac{r}{4}+\tfrac{r-U_*}{4}+\tfrac{P}{4}=\tfrac{19k+6r}{8}-1<1,$$

using $\alpha+\beta=r-\gamma\le r-U_*$ and $\beta\le P$. Its second term is

$$\tfrac{3k}{4}+\tfrac{3\beta}{4}+\alpha+\tfrac{\gamma}{4}=\tfrac{3k}{4}+\tfrac{r}{4}+\tfrac{\beta}{2}+\tfrac{3\alpha}{4}\le\tfrac{3k}{4}+\tfrac{r}{4}+\tfrac{5}{8}(\alpha+\beta)\le\tfrac{21k+7r}{8}-\tfrac{5}{4}<1,$$

using $\alpha\le\beta$ and again $\alpha+\beta\le r-U_*=r-2+3k$. □

The lemma has also been checked numerically on a grid.

6.3 The balanced pairing

Lemma 57 (Localisation). *Let $c\in C_c^\infty(\mathbb{R}^3)$ and $I_c=\{|a|:\ a\in[0,\infty)^3\cap\operatorname{supp}c\}$. Then Lemma 50 holds with W_ψ and I_ψ replaced by P_c and I_c.*

Proof. The coefficient of P_c at n is $\sum_{n_1n_2n_3=n}c(\ell(n))\ll\tau_3(n)$, and it vanishes unless $\log n/L\in I_c$. These are the only properties of M_ψ used in the proof of Lemma 50. □

Proposition 58 (Balanced pairing). *Let $c\in C_c^\infty(\mathbb{R}^3)$ be real with $\operatorname{supp}c\subset\mathcal{R}^\sharp$, and let $b\in C_c^\infty(\mathbb{R}^3)$. Uniformly for Φ in bounded subsets of C_c^∞ with a fixed compact support in $(1,2)$,*

$$\langle P_b,P_c\rangle_\Phi=a_3TL^9\Big(\int\Phi\Big)\int_\Omega h_bh_c+o(TL^9),$$

where h_b is the profile of the restriction of b to $[0,1]^3$.

Here b may be complex-valued. Lemma 6 and its proof hold verbatim for complex-valued continuous F, with $c_\lambda(F)$ complex; in part (iii) the first limit is then $\int_\Omega h_F\overline{h_G}$.

Reduction to elementary sources. Fix c; implied constants may depend on c and on the support of Φ. Let $\varphi_1,\phi\in C_c^\infty((1,2))$ with $\varphi_1=1$ on a neighbourhood of I_c and $\phi=1$ on a neighbourhood of $\operatorname{supp}\varphi_1$. Put $b_1(a)=\varphi_1(|a|)b(a)$.

The coefficients of $P_b - P_{b_1}$ vanish at every N with $\log N/L$ in a neighbourhood of I_c, so $\langle P_b - P_{b_1}, P_c\rangle_\Phi = O(T^{-A})$ by Lemma 57. On $[0,\infty)^3 \cap \operatorname{supp} b_1$ we have $\phi(|a|) = 1$, so (27) gives $P_{b_1} = (2\pi)^{-3}\int \widehat{b_1}(\xi) P_{-i\xi,\phi}\,\mathrm{d}\xi$. For $\boldsymbol{u} \in (i\mathbb{R})^3$, Cauchy–Schwarz with respect to a smooth non-negative majorant of $|\Phi|$, Lemma 33 and Lemma 31(ii) give

$$|\langle P_{\boldsymbol{u},\phi}, P_c\rangle_\Phi| \ll (1+|\boldsymbol{u}|)^3 T L^9.$$

We shall show that for every $\boldsymbol{u} \in (i\mathbb{R})^3$ with distinct coordinates

$$\langle P_{\boldsymbol{u},\phi}, P_c\rangle_\Phi = a_3 T L^9 \Big(\int \Phi\Big) \int_\Omega h_{e^{-\boldsymbol{u}\cdot a}} h_c + o(TL^9). \tag{42}$$

Granting this, dominated convergence in ξ (the integrand is $\ll |\widehat{b_1}(\xi)|(1+|\xi|)^3 TL^9$, and almost every ξ has distinct coordinates) and Fubini's theorem in (2), with $(2\pi)^{-3}\int \widehat{b_1}(\xi) e^{i\xi\cdot a}\,\mathrm{d}\xi = b_1(a)$, give $\langle P_{b_1}, P_c\rangle_\Phi = a_3 TL^9 (\int\Phi)\int_\Omega h_{b_1} h_c + o(TL^9)$. Finally h_c vanishes unless $r \in I_c$, because $\kappa(x,y,z;a) = 0$ unless $|a| = r$, and there $h_{b_1} = \varphi_1(r) h_b = h_b$. This proves Proposition 58, given (42).

The rest of this subsection proves (42). Fix $\boldsymbol{u} \in (i\mathbb{R})^3$ with distinct coordinates, put $\boldsymbol{\alpha} = \boldsymbol{u}/L$ and $r_0 = 1 + \max_k |u_k|$, and let

$$\rho_- = \min I_c > 1, \qquad \rho_+ = \max I_c < \tfrac{15}{14},$$
$$\kappa_+ = \max\{a_1 + a_2 :\ a \in \operatorname{supp} c\} < \tfrac{11}{21}.$$

For $q \geq 1$ and $a_3 \in \mathbb{R}$ put $\gamma_q(a_3) = \sum_{d_1 d_2 = q} c(\log d_1/L, \log d_2/L, a_3)$, so that $P_c(s) = \sum_{q,m} \gamma_q(\log m/L)(qm)^{-s}$. Each γ_q is real and smooth, $\partial^j \gamma_q \ll_j \tau(q)$, and $\gamma_q(a_3) \neq 0$ implies $\log q/L \leq \kappa_+$ and $\log q/L + a_3 \in I_c$. With $\widehat{\Phi}(\xi) = \int \Phi(x) e(-x\xi)\,\mathrm{d}x$ let

$$W_{q,h}(y) = \frac{1}{y}\,\gamma_q\Big(\frac{\log(y/q)}{L}\Big)\,\widehat{\Phi}\Big(\frac{Th}{2\pi y}\Big) \qquad (q \geq 1,\ h \in \mathbb{Z},\ y > 0),$$

a smooth function supported in $[T^{\rho_-}, T^{\rho_+}]$, and for $g \mid q$ let $Z_{q,g}(s) = \sum_{(n,q)=g} b_{\boldsymbol{u}}(n) n^{-s}$. We write $\mathcal{I} = \langle P_{\boldsymbol{u},\phi}, P_c\rangle_\Phi$, and $\eta > 0$ denotes a small constant depending on c, which may decrease from one lemma to the next.

Lemma 59 (Truncation). *$\mathcal{I} = T\sum_{q\geq 1}\sum_{h\in\mathbb{Z}} S_{q,h} + O(T^{1-\eta})$, where $S_{q,h} = \sum_{n \equiv h\ (q)} b_{\boldsymbol{u}}(n) W_{q,h}(n)$.*

Proof. Expanding both polynomials and integrating in t,

$$\mathcal{I} = T\sum_{n}\sum_{q,m} b_{\boldsymbol{u}}(n)\phi\Big(\frac{\log n}{L}\Big)\gamma_q\Big(\frac{\log m}{L}\Big)(nqm)^{-1/2}\,\widehat{\Phi}\Big(\frac{T}{2\pi}\log\frac{n}{qm}\Big),$$

a finite sum with $n \le T^2$ and $qm \le T^{\rho_+}$. Fix $\epsilon > 0$. Since $\widehat{\Phi}(\xi) \ll_A |\xi|^{-A}$, the terms with $|\log(n/qm)| > T^{-1+\epsilon}$ contribute $O(T^{-A})$. In the remaining terms put $h = n - qm$, so $|h| \le 2nT^{-1+\epsilon}$; as $\log(qm)/L \in I_c$ we have $\phi(\log n/L) = 1$. Then $(nqm)^{-1/2} = n^{-1}(1 + O(T^{-1+\epsilon}))$, $\frac{T}{2\pi}\log\frac{n}{qm} = \frac{Th}{2\pi n} + O(T^{-1+2\epsilon})$ and $\gamma_q(\log m/L) = \gamma_q(\log(n/q)/L) + O(\tau(q)T^{-1+\epsilon})$. For each $n \ll T^{\rho_+}$ and each h there are at most $\tau(n-h) \ll T^{\epsilon}$ moduli $q \mid n-h$, so these replacements cost

$$\ll T\cdot T^{-1+3\epsilon}\sum_{n\ll T^{\rho_+}}\frac{\tau_3(n)}{n}\cdot nT^{-1+\epsilon} \ll T^{\rho_+-1+5\epsilon}.$$

Since $q \le T^{\kappa_+} < n/2$, every pair (q,h) with $q \mid n-h$ and $|h| \le 2nT^{-1+\epsilon}$ comes from a unique $m = (n-h)/q \ge 1$. Finally, adding the terms with $q \mid n-h$ and $|h| > 2nT^{-1+\epsilon}$ costs $O(T^{-A})$, by the decay of $\widehat{\Phi}$; in the new terms $\gamma_q(\log(n/q)/L) \ne 0$ still forces $\log n/L \in I_c$. As $\rho_+ - 1 < 1$, this proves the lemma after rearranging the absolutely convergent sum. □

Lemma 60 (Equidistribution). $T\sum_{q}\sum_{h} S_{q,h} = T\sum_{q}\sum_{h}\mathfrak{m}_{q,h} + O(T^{1-\eta})$, *where, with* $g = (h,q)$,

$$\mathfrak{m}_{q,h} = \frac{1}{\varphi(q/g)}\sum_{(n,q)=g} b_{\boldsymbol{u}}(n)W_{q,h}(n).$$

Proof. For $h = 0$, $S_{q,0} = \mathfrak{m}_{q,0}$. Let $h \ne 0$. Fix $V_0 \in C_c^\infty([\frac12, 2])$ with $\sum_{i\in\mathbb{Z}} V_0(y/2^i) = 1$ for $y > 0$; for integers $n_j \ge 1$ only $i \ge -1$ contribute. Then $S_{q,h} - \mathfrak{m}_{q,h} = \sum_{U,V,W} D_q(h; w_{q,h}^{UVW})$, in the notation of Proposition 55, where U, V, W run over $\{2^i : i \ge -1\}$ and

$$w_{q,h}^{UVW}(y) = V_0\Big(\frac{y_1}{U}\Big)V_0\Big(\frac{y_2}{V}\Big)V_0\Big(\frac{y_3}{W}\Big)\prod_{j=1}^{3} y_j^{-u_j/L}\; W_{q,h}(y_1y_2y_3),$$

which is supported in $[\frac{U}{2}, 2U]\times[\frac{V}{2}, 2V]\times[\frac{W}{2}, 2W]$. As $W_{q,h}$ is supported in $[T^{\rho_-}, T^{\rho_+}]$, only $O(L^3)$ triples contribute, each with $X := UVW \in [T^{\rho_-}/8, 8T^{\rho_+}]$; and only $q \le T^{\kappa_+}$ contribute, in $O(L)$ dyadic ranges $Q < q \le 2Q$.

Since $(y_j\partial_j)F(y_1y_2y_3) = ((y\partial_y)F)(y_1y_2y_3)$, the Leibniz rule gives $|(y_1\partial_1)(y_2\partial_2)(y_3\partial_3)w_{q,h}^{UVW}| \ll (1+|\boldsymbol{u}|)^3 \max_{i\le 3}\sup_{y\asymp X}|(y\partial_y)^i W_{q,h}(y)|$. Now $(y\partial_y)^j\widehat{\Phi}(\lambda/y)$ is a combination of the functions $(\lambda/y)^l\widehat{\Phi}^{(l)}(\lambda/y)$, $l\le j$, and $\widehat{\Phi}$ is a Schwartz function; hence, with $H = X/T$, Proposition 55 applies with

$$M \ll_B (1+|\boldsymbol{u}|)^3 T^{\epsilon} X^{-1}(1+|h|/H)^{-B}.$$

It gives $\sum_{Q<q\le 2Q}|D_q(h;w_{q,h}^{UVW})| \ll (1+|\boldsymbol{u}|)^3 T^{2\epsilon}X^{-1}(1+|h|/H)^{-B}(2+|h|)^{\epsilon}\, Q\min_\pi\min_i E_i$. As $H \ge T^{\rho_- -1}/8 \ge 1$, summing over $h\neq 0$ gives a factor $\ll H^{1+\epsilon}$, so the contribution of one dyadic choice of (Q,U,V,W) to $T\sum_q\sum_h(S_{q,h}-\mathfrak{m}_{q,h})$ is

$$\ll (1+|\boldsymbol{u}|)^3 T^{3\epsilon}\, Q \min_\pi \min_i E_i(\pi(U^+,V^+,W^+)) \le 3(1+|\boldsymbol{u}|)^3 T^{3\epsilon} T^{\theta(k;a)},$$

where $Q = T^k$ and $(U^+,V^+,W^+) = (T^{a_1},T^{a_2},T^{a_3})$. If this choice contributes at all, there are $Q<q\le 2Q$ and $n\asymp X$ with $\gamma_q(\log(n/q)/L)\neq 0$, so $(\log q/L, \log n/L) = (a_1'+a_2', |a'|)$ for some $a'\in\operatorname{supp} c$. Hence $(k,\log X/L)$ lies within $O(1/L)$ of the compact set $K_c = \{(a_1+a_2,|a|) : a\in\operatorname{supp} c\}\subset\mathcal{O}$; and $|a_1+a_2+a_3-\log X/L|\le 3\log 2/L$, because $U\le U^+\le 2U$. Applying Lemma 56 with K a compact neighbourhood of K_c in $\mathcal{O}$, and using that θ is nondecreasing in k, we get $\theta(k;a)\le 1-\eta_0$ for L large. Summing over the $O(L^4)$ dyadic choices and taking $\epsilon<\eta_0/4$ proves the lemma. □

Lemma 61 (Polar part). *Let $\mathcal{C}$ be the positively oriented circle $|s-1| = r_0/L$, and $\widetilde{W}(s) = \int_0^\infty W(y)y^{s-1}\,\mathrm{d}y$. Then $T\sum_q\sum_h \mathfrak{m}_{q,h} = \mathcal{M} + O(T^{1-\eta})$, where*

$$\mathcal{M} = T\sum_q\sum_h \frac{1}{\varphi(q/g)}\cdot\frac{1}{2\pi i}\oint_{\mathcal{C}}\widetilde{W}_{q,h}(s)Z_{q,g}(s)\,\mathrm{d}s.$$

Proof. For $p\mid q$ let $J_p = \{v_p(g)\}$ if $v_p(g)<v_p(q)$ and $J_p = \{j : j\ge v_p(q)\}$ if $v_p(g) = v_p(q)$; then $(n,q) = g$ if and only if $v_p(n)\in J_p$ for every $p\mid q$. Hence

$$Z_{q,g}(s) = \prod_{k=1}^{3}\zeta(s+\alpha_k)\,E_{q,g}(s),$$

$$E_{q,g}(s) = \prod_{p\mid q}\Big[\prod_{k=1}^{3}(1-p^{-s-\alpha_k})\sum_{j\in J_p} b_{\boldsymbol{u}}(p^j)p^{-js}\Big].$$

Each factor of $E_{q,g}$ is entire: if J_p is infinite the sum equals $\prod_k(1-p^{-s-\alpha_k})^{-1}$ minus a finite sum. As $|b_{\boldsymbol{u}}(p^j)|\le\binom{j+2}{2}$, for $\Re s\ge\frac34$ each factor is at most $C(v_p(g)+1)^2p^{-3v_p(g)/4}$, so $|E_{q,g}(s)|\ll_\epsilon q^\epsilon g^{-3/4}$.

By Mellin inversion $\sum_{(n,q)=g} b_{\boldsymbol{u}}(n) W_{q,h}(n) = \frac{1}{2\pi i}\int_{(2)} \widetilde{W}_{q,h}(s) Z_{q,g}(s)\,\mathrm{d}s$. Write $W_{q,h}(y) = y^{-1}w(y)$. For $j \ge 0$, integration by parts gives $\widetilde{W}_{q,h}(s) = (1-s)^{-j}\int (y\partial_y)^j w(y)\, y^{s-2}\,\mathrm{d}y$, and $|(y\partial_y)^j w(y)| \ll_{j,B} \tau(q)(1+|h|/H_+)^{-B}$ with $H_+ = T^{\rho_+ - 1}$, because $T|h|/y \ge |h|/H_+$ on the support. Taking $j=0$ for $|s-1| \le \frac18$ and $j=2$ otherwise, and using $|s| \le 9|s-1|$ when $|s-1| \ge \frac18$, we get for $\frac34 \le \Re s \le 2$

$$|\widetilde{W}_{q,h}(s)| \ll_B \tau(q)(1+|s|)^{-2}(1+|h|/H_+)^{-B}\int_{T^{\rho_-}}^{T^{\rho_+}} y^{\Re s - 2}\,\mathrm{d}y. \tag{43}$$

The integral is at most $4T^{-\rho_-/4}$ on $\Re s = \frac34$ and at most T^{ρ_+} for $\frac34 \le \Re s \le 2$. In particular $|\widetilde{W}_{q,h}(\frac34 + it)| \ll \tau(q)(1+|t|)^{-2}(1+|h|/H_+)^{-B}T^{-\rho_-/4}$, while $Z_{q,g}(s)$ grows at most polynomially in $|\Im s|$, like $|\Im s|^{3/4+\epsilon}$, uniformly in $\frac34 \le \Re s \le 2$ away from the poles (convexity bound and $|E_{q,g}| \ll q^{\epsilon}$). So we may move the line of integration to $\Re s = \frac34$. The poles crossed are at $s = 1-\alpha_k$, inside $\mathcal{C}$ as $|\alpha_k| < r_0/L$, and $\mathcal{C}$ lies in $\Re s > \frac34$; so the residues sum to the integral over $\mathcal{C}$. By the convexity bound $\prod_k |\zeta(\frac34 + it + \alpha_k)| \ll (1+|t|)^{1/2}(1+|\boldsymbol{u}|)$. The integrals on $\Re s = \frac34$ therefore contribute to $T\sum_q \sum_h \mathfrak{m}_{q,h}$ at most

$$\ll (1+|\boldsymbol{u}|)T^{1-\rho_-/4+\epsilon}\sum_{q \le T^{\kappa_+}}\sum_{g\mid q}\frac{g^{-3/4}}{\varphi(q/g)}\sum_{(h,q)=g}(1+|h|/H_+)^{-B}.$$

The innermost sum is $\ll 1 + H_+/g$, and $\varphi(q/g)^{-1} \ll T^{\epsilon}g/q$, so the sum over $g \mid q$ is $\ll T^{2\epsilon}(q^{-3/4} + H_+ q^{-1})$ and the sum over q is $\ll T^{3\epsilon}(T^{\kappa_+/4} + H_+)$. The total is

$$\ll (1+|\boldsymbol{u}|)T^{5\epsilon}\big(T^{1-(\rho_- - \kappa_+)/4} + T^{\rho_+ - \rho_-/4}\big),$$

and both exponents are below 1, because $\kappa_+ < 1 < \rho_-$ and $\rho_+ - \frac{\rho_-}{4} < \frac{15}{14} - \frac14$. □

We now evaluate $\mathcal{M}$. For a prime p and integers $a, j, m \ge 0$ put

$$f_p(a,j,m) = \begin{cases} 1, & a \le j+m, \\ -1/(p-1), & a = j+m+1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\varepsilon_p(a,j;s) = \prod_{k=1}^{3}\big(1-p^{-s-\alpha_k}\big)\sum_{m\ge 0} d_{\boldsymbol{\alpha}}(p^m)p^{-ms}f_p(a,j,m),$$

where $d_{\boldsymbol{\alpha}}(n) = \sum_{n_1n_2n_3=n} \prod_k n_k^{-\alpha_k} = b_{\boldsymbol{u}}(n)$. Note that $\varepsilon_p(a,j;s) = 1$ when $a \le j$. For $\boldsymbol{v} = (v_1, v_2)$ put $\sigma_{\boldsymbol{v}}(n) = \sum_{d_1d_2=n} d_1^{-v_1} d_2^{-v_2}$, and for $(\omega, \boldsymbol{v}, w) \in \mathbb{C}^4$ let

$$\mathcal{S}_p = \sum_{a,j\ge 0} \varepsilon_p(a,j;1+\omega)\, \sigma_{\boldsymbol{v}}(p^a)\, p^{a\omega - j(1+\omega+w)}, \qquad \mathcal{S} = \prod_p \mathcal{S}_p.$$

Lemma 62 (Euler product)**.** *Suppose that ω, v_1, v_2, w and $\alpha_1, \alpha_2, \alpha_3$ have real parts in $[-\frac{1}{20}, \frac{1}{20}]$.*

(i) *The series $\mathcal{S}_p$ converges absolutely, and*

$$\mathcal{S} = \frac{\zeta(1+\omega+w)\prod_{i=1}^2 \zeta(1+v_i+w)\prod_{i=1}^2\prod_{k=1}^3 \zeta(1+v_i+\alpha_k)}{\prod_{i=1}^2 \zeta(1-\omega+v_i)}\, \mathfrak{S},$$

where $\mathfrak{S} = \prod_p \mathfrak{S}_p$ with $\mathfrak{S}_p = 1 + O(p^{-3/2})$ uniformly; so $\mathfrak{S}$ is holomorphic and bounded, and the identity holds wherever the left side converges.

(ii) *Let $0 < \delta \le \frac{1}{20}$ and suppose in addition that $\Re(v_i - \omega)$, $\Re(\omega + w)$, $\Re(v_i + w)$ and $\Re(v_i + \alpha_k)$ are at least δ for all i, k. Then*

$$\sum_{q,J\ge 1} \Big| \sigma_{\boldsymbol{v}}(q)\, q^{\omega} J^{-1-\omega-w} \prod_p \varepsilon_p\big(v_p(q), v_p(J); 1+\omega\big) \Big| \ll \delta^{-11},$$

and the sum without absolute values equals $\mathcal{S}$.

(iii) $\mathfrak{S}(0) = a_3$.

Proof. Expand $\varepsilon_p(a,j;1+\omega)\sigma_{\boldsymbol{v}}(p^a)p^{a\omega - j(1+\omega+w)}$, using $(p-1)^{-1} = \sum_{l\ge 1} p^{-l}$, as a sum of monomials $p^{-d}p^{-\ell}$, where $d \ge 0$ is an integer, called the degree, and ℓ is a linear form in the shifts. The term (a,j) only produces monomials of degree at least $\max(a,j)$: for $a \le j$ this is clear, and for $a > j$ the factor f_p forces $m \ge a - j - 1$, with the extra factor $(p-1)^{-1}$ if $m = a - j - 1$. Each monomial of degree d has $|\Re\ell| \le \frac{4d}{20}$, since $a \le d$ and $j + m \le d$, We expand completely before collecting terms. Then every monomial carries a coefficient of modulus at most an absolute constant: a product of signs from μ and of the finitely many coefficients of the numerator of f_p, all other expansions (the geometric series for $(p-1)^{-1}$, the divisor-type sums $\sigma_{\boldsymbol{v}}$ and $d_{\boldsymbol{\alpha}}$) having coefficients 1. A monomial of degree d is determined by finitely many choices in the numerator together with a composition of d into a bounded number of non-negative exponents, so there are $O((1+d)^C)$ of them. Hence the monomials of degree d contribute $\ll (1+d)^C p^{-4d/5}$ in

absolute value, and $\mathcal{S}_p$ converges absolutely. The monomials of degree 0 give 1, and those of degree 1 come from $(a,j)\in\{(0,1),(1,1),(1,0)\}$:

$$F_p = p^{-1-\omega-w} + \sum_i p^{-1-v_i-w} + \sum_{i,k} p^{-1-v_i-\alpha_k} - \sum_i p^{-1+\omega-v_i};$$

here $\varepsilon_p(1,0;1+\omega) = \sum_k p^{-1-\omega-\alpha_k} - p^{-1} + (\text{degree} \geq 2)$ and $\sigma_{\boldsymbol{v}}(p)p^{\omega} = \sum_i p^{\omega-v_i}$. So $\mathcal{S}_p = 1 + F_p + O(p^{-8/5})$, while the local factor of the displayed quotient of zeta functions is $1 + F_p + O(p^{-8/5})$ as well. Hence $\mathfrak{S}_p$, defined as $\mathcal{S}_p$ divided by that local factor, is $1 + O(p^{-8/5})$, which proves (i).

For (ii), the summand is multiplicative in the pair (q,J): it is the product over all primes p of the term $(a,j) = (v_p(q), v_p(J))$ of $\mathcal{S}_p$, and the term $(0,0)$ equals 1. So the sum of absolute values is at most $\prod_p(1 + \sum_{\text{monomials}\neq 1} |p^{-d-\ell}|)$. The monomials of degree 1 are the eleven terms of F_p, whose absolute values are $p^{-1-\Re(\omega+w)}$, $p^{-1-\Re(v_i+w)}$, $p^{-1-\Re(v_i+\alpha_k)}$ and $p^{-1-\Re(v_i-\omega)}$, all at most $p^{-1-\delta}$. The monomials of degree $d \geq 2$ contribute $\ll (1+d)^C p^{-4d/5}$, whose sum over $d \geq 2$ and p converges. So the product is $\ll \exp(11\sum_p p^{-1-\delta}) \ll \delta^{-11}$, since $\sum_p p^{-1-\delta} \leq \log(1/\delta) + O(1)$. Absolute convergence allows the multiplicative rearrangement, so the sum is $\mathcal{S}$.

For (iii), at zero shifts $\sigma_{\mathbf{0}}(p^a) = a+1$ and $d_{\mathbf{0}}(p^m) = \binom{m+2}{2}$. With $X = 1/p$ and $r = j+m$, since $\sum_{m\leq r}\binom{m+2}{2} = \binom{r+3}{3}$,

$$(1-X)^{-3}\mathcal{S}_p(0) = \sum_{r\geq 0}\binom{r+3}{3}X^r\Big[\binom{r+2}{2} - \frac{X}{1-X}(r+2)\Big] = \frac{1+4X+X^2}{(1-X)^6},$$

as in the proof of Proposition 22. Removing the local factors of the seven net zeta factors, $\mathfrak{S}_p(0) = (1-X)^7\mathcal{S}_p(0) = (1-X)^4(1+4X+X^2)$, which is the local factor of a_3 by the computation in the proof of Lemma 28. □

The zero-shift identity in (iii) has also been checked by computer as an identity of power series in X to order 16, and the first-order statement in (i) exactly: as polynomials in p^{-1} whose coefficients are Laurent polynomials in $p^{-\omega}$, p^{-v_i}, p^{-w} and $p^{-\alpha_k}$, the local factors $\mathcal{S}_p$ and that of the zeta quotient have the same terms of degree 0 and 1 (the eleven terms of F_p). It also evaluates their ratio at $p = 101$ and $p = 409$ at one point with all real parts in $[-\frac{1}{20}, \frac{1}{20}]$, in floating point.

Lemma 63 (Poisson summation)**.** *For $\beta \in \mathbb{C}^3$ let*

$$\mathcal{L}c(\beta) = \int_{\mathbb{R}^3} c(a)e^{\beta_1 a_1+\beta_2 a_2+\beta_3(1-a_3)}\,\mathrm{d}a,$$

and let $\widetilde{\Phi}(s)=\int\Phi(x)x^{s-1}\,\mathrm{d}x$. *Then*

$$\mathcal{M}=\frac{T}{(2\pi i)^4}\oint_{|\omega|=r_0/L}\int_{(2r_0)^3}\mathcal{L}c(\beta)e^{-\beta_3}\Big(\frac{T}{2\pi}\Big)^{\omega+w}\\ \times\widetilde{\Phi}(1+\omega+w)\prod_{k=1}^{3}\zeta(1+\omega+\alpha_k)\,\mathcal{S}\,\mathrm{d}\beta\,\mathrm{d}\omega,$$

where $\int_{(2r_0)^3}$ *denotes integration over* $\Re\beta_1=\Re\beta_2=\Re\beta_3=2r_0$, $\mathcal{S}=\mathcal{S}(\omega;\boldsymbol{v},w)$ *with* $\boldsymbol{v}=(\beta_1,\beta_2)/L$ *and* $w=\beta_3/L$, *and the integral converges absolutely.*

Proof. Poisson summation in h. The indicator of $(h,q)=g$ is q-periodic in h, and $\sum_{h \bmod q,\,(h,q)=g}e(-hJ/q)=c_{q/g}(J)$; hence $\mathbf{1}_{(h,q)=g}=q^{-1}\sum_{J \bmod q}c_{q/g}(J)e(hJ/q)$. For $y>0$ let $F(x)=\widehat{\Phi}(Tx/(2\pi y))$, so that $\widehat{F}(\xi)=\frac{2\pi y}{T}\Phi(-\frac{2\pi y\xi}{T})$. By Poisson summation, and since $c_{q/g}$ is even and q-periodic,

$$\sum_{(h,q)=g}\widehat{\Phi}\Big(\frac{Th}{2\pi y}\Big)=\frac{1}{q}\sum_{J\in\mathbb{Z}}c_{q/g}(J)\widehat{F}\Big(\frac{J}{q}\Big)=\frac{2\pi y}{qT}\sum_{J\ge1}c_{q/g}(J)\,\Phi\Big(\frac{2\pi yJ}{qT}\Big),$$

the terms with $J\le0$ vanishing because Φ is supported in $(1,2)$. The sum over h in $\mathcal{M}$ is absolutely convergent uniformly for $s\in\mathcal{C}$ and y in the support, so we may perform it inside the integrals. This gives

$$\mathcal{M}=\sum_{q}\sum_{J\ge1}\frac{2\pi}{q}\cdot\frac{1}{2\pi i}\oint_{\mathcal{C}}\int_0^\infty y^{s-1}\gamma_q\Big(\frac{\log(y/q)}{L}\Big)\Phi\Big(\frac{2\pi yJ}{qT}\Big)\,\mathrm{d}y\;\mathcal{Z}_{q,J}(s)\,\mathrm{d}s,$$

where

$$\mathcal{Z}_{q,J}(s)=\sum_{g\mid q}\frac{c_{q/g}(J)}{\varphi(q/g)}Z_{q,g}(s),$$

with only finitely many (q,J), because $\Phi(2\pi yJ/(qT))\neq0$ forces $1<2\pi yJ/(qT)<2$, so $J<qT/(\pi y)<2qT^{1-\rho_-}$.

The Ramanujan factor. By Hölder's evaluation

$$c_l(J)=\mu\big(l/(l,J)\big)\varphi(l)/\varphi\big(l/(l,J)\big)$$

[20, §8.4]. If $(n,q)=g$ and $l=q/g$, then $(q,Jn)=g(l,J)$, because $(l,n/g)=1$; so $c_{q/g}(J)/\varphi(q/g)=f(q,Jn)$ with $f(q,N)=\mu(q/(q,N))/\varphi(q/(q,N))$. Hence, for $\Re s>1$ and then by continuation,

$$\mathcal{Z}_{q,J}(s)=\sum_{n\ge1}b_{\boldsymbol{u}}(n)n^{-s}f(q,Jn)=\prod_{k=1}^{3}\zeta(s+\alpha_k)\prod_{p}\varepsilon_p\big(v_p(q),v_p(J);s\big),$$

because $f(q,Jn)=\prod_p f_p(v_p(q),v_p(J),v_p(n))$; the product over p is finite, as $\varepsilon_p(0,j;s)=1$.

Change of variables. Substitute $y=qTx/(2\pi J)$ and $s=1+\omega$. Then $\frac{2\pi}{q}y^{s-1}\,\mathrm{d}y=T(\frac{T}{2\pi})^{\omega}q^{\omega}J^{-1-\omega}x^{\omega}\,\mathrm{d}x$, and

$$\mathcal{M}=\frac{T}{2\pi i}\oint_{|\omega|=r_0/L}\Big(\frac{T}{2\pi}\Big)^{\omega}\sum_{q,J}q^{\omega}J^{-1-\omega}\mathcal{Z}_{q,J}(1+\omega)\\ \times\int x^{\omega}\Phi(x)\gamma_q\Big(\frac{\log(Tx/(2\pi J))}{L}\Big)\,\mathrm{d}x\,\mathrm{d}\omega.$$

Laplace inversion. Since $c\in C_c^\infty$, $\mathcal{L}c$ is entire and $|\mathcal{L}c(\beta)|\ll_N(1+|\beta|)^{-N}$ on $\Re\beta_i=2r_0$, and $c(a)=(2\pi i)^{-3}\int_{(2r_0)^3}\mathcal{L}c(\beta)e^{-\beta_1a_1-\beta_2a_2-\beta_3(1-a_3)}\,\mathrm{d}\beta$. With $a_3=\log(Tx/(2\pi J))/L$ this gives

$$\gamma_q\Big(\frac{\log(Tx/(2\pi J))}{L}\Big)=\frac{1}{(2\pi i)^3}\int_{(2r_0)^3}\mathcal{L}c(\beta)\,\sigma_{\boldsymbol{v}}(q)\,e^{-\beta_3}\Big(\frac{Tx}{2\pi J}\Big)^{w}\,\mathrm{d}\beta.$$

Insert this and move the sums over q,J inside the β-integral. On the contours $\Re\omega\in[-r_0/L,r_0/L]$ and $\Re v_i=\Re w=2r_0/L$, while the α_k are imaginary; so the hypotheses of Lemma 62(ii) hold with $\delta=r_0/L$, for L large. Together with $|\zeta(1+\omega+\alpha_k)|\ll L$ on $\mathcal{C}$ (as $|\omega+\alpha_k|\ge 1/L$) and $|x^w|\le 4$, this justifies the interchange, and Lemma 62(ii) identifies $\sum_{q,J}\sigma_{\boldsymbol{v}}(q)q^{\omega}J^{-1-\omega-w}\mathcal{Z}_{q,J}(1+\omega)$ with $\prod_k\zeta(1+\omega+\alpha_k)\mathcal{S}$. The x-integral is $\widetilde{\Phi}(1+\omega+w)$. □

Lemma 64 (Limit). *$\mathcal{M}=a_3TL^9(\int\Phi)\Lambda+o(TL^9)$, where, with $\boldsymbol{\sigma}=(u_1,u_2,u_3,\beta_3)$,*

$$\Lambda=\frac{1}{(2\pi i)^4}\oint_{|\nu|=r_0}\int_{(2r_0)^3}\mathcal{L}c(\beta)\,\frac{e^{\nu}\prod_{i=1}^{2}(\beta_i-\nu)}{\prod_{l=1}^{4}(\nu+\sigma_l)\prod_{i=1}^{2}\prod_{l=1}^{4}(\beta_i+\sigma_l)}\,\mathrm{d}\beta\,\mathrm{d}\nu.$$

Proof. Substitute $\omega=\nu/L$ in Lemma 63. Since $\zeta(1+s)-1/s$ is entire, $L^{-1}\zeta(1+z/L)\to 1/z$ for each $z\ne 0$. Using Lemma 62(i), the integrand of Lemma 63, multiplied by $\mathrm{d}\omega=\mathrm{d}\nu/L$ and divided by TL^9, converges pointwise to $a_3(\int\Phi)$ times the integrand of Λ: the zeta factors give $L^{3+1+2+6-2}=L^{10}$ times the corresponding linear factors, $\mathfrak{S}\to\mathfrak{S}(0)=a_3$, $(\frac{T}{2\pi})^{\omega+w}e^{-\beta_3}\to e^{\nu}$ and $\widetilde{\Phi}(1+\omega+w)\to\int\Phi$; and $\prod_k(\nu+u_k)(\nu+\beta_3)=\prod_l(\nu+\sigma_l)$, $\prod_i(\beta_i+\beta_3)\prod_{i,k}(\beta_i+u_k)=\prod_{i,l}(\beta_i+\sigma_l)$. For dominated convergence we bound the factors on the contours, uniformly in L. On $|\nu|=r_0$ we have $|\nu+u_k|\ge 1$, so $|L^{-1}\zeta(1+(\nu+u_k)/L)|\le 2$. The other zeta factors in the numerator

have arguments $1+z/L$ with $\Re z \ge r_0$, so $|L^{-1}\zeta(1+z/L)| \le L^{-1}\zeta(1+\Re z/L) \le 2/r_0$. For the denominator, $\Re(\beta_i-\nu)\ge r_0$ and $|L/\zeta(1+z/L)| \ll (1+|z|)^2$ for $\Re z \ge r_0$, as in the proof of Proposition 32. Finally $\mathfrak{S}$ is bounded, $|(\frac{T}{2\pi})^{\omega+w}e^{-\beta_3}| \le e^{2r_0}$ and $|\widetilde{\Phi}(1+\omega+w)| \le 4\int|\Phi|$. So the normalised integrand is $\ll |\mathcal{L}c(\beta)|(1+|\beta|)^4$, which is integrable. □

Lemma 65 (Four-variable Haar identity). *Let $A, B \ge 0$ be integers with $A+B \le N$. For $X \in \mathbb{C}^4$ with distinct coordinates,*

$$\Big\langle \prod_{i=1}^{4} \det(I+X_iU),\ D\,e_Ae_B \Big\rangle_N = \sum_{i=1}^{4} \frac{X_i^{N+3}h_A(X_{\neq i})h_B(X_{\neq i})}{\prod_{l\neq i}(X_i-X_l)},$$

where h_A is the complete homogeneous symmetric polynomial of degree A and $X_{\neq i} \in \mathbb{C}^3$ omits X_i. If $|X_i| \le 1$ for all i, the left side has absolute value at most $(N+1)(N+3)^6$.

Proof. By the dual Cauchy identity [10, I.(4.3′)], $\prod_{i=1}^4 \det(I+X_iU) = \sum_\lambda s_{\lambda'}(X)s_\lambda(U)$, over the λ with $\ell(\lambda) \le N$ and $\lambda_1 \le 4$. By (S2), $e_Ae_B = \sum_{y=0}^{\min(A,B)} s_\nu$ with $\nu' = (A+B-y, y)$, and $Ds_\nu = s_{\nu+(1^N)}$, a Schur polynomial whose column lengths are $(N, A+B-y, y)$; this is a partition because $A+B \le N$. By (S1) the left side is $\sum_y s_{(N,A+B-y,y,0)}(X)$. For a partition μ with at most four parts, $s_\mu(X) = \det(X_i^{\mu_l+4-l})/\det(X_i^{4-l})$. For $\mu = (N, j, A+B-j, 0)$ expand the numerator along the column $l=1$, of exponent $N+3$. The minor omitting row i is $V(X_{\neq i})s_{(j,A+B-j)}(X_{\neq i})$, where $V(Y) = \prod_{a<b}(Y_a-Y_b)$, and $\det(X_i^{4-l}) = V(X) = (-1)^{i-1}V(X_{\neq i})\prod_{l\neq i}(X_i-X_l)$. Hence

$$s_\mu(X) = \sum_{i=1}^{4} \frac{X_i^{N+3}s_{(j,A+B-j)}(X_{\neq i})}{\prod_{l\neq i}(X_i-X_l)},$$

and summing over $\max(A,B) \le j \le A+B$ with Pieri's rule $h_Ah_B = \sum_j s_{(j,A+B-j)}$ [10, I.(5.16)] gives the identity. For the bound, s_μ has non-negative coefficients, so $|s_\mu(X)| \le s_\mu(1,1,1,1) = \prod_{a<b}\frac{\mu_a-\mu_b+b-a}{b-a} \le (N+3)^6$, and there are at most $N+1$ values of y. □

As a finite consistency check, we compared the two sides in exact rational arithmetic for five triples (N,A,B) at one rational point $X \in \mathbb{Q}^4$; the proof above gives the general identity.

Lemma 66 (Kernel identity). *Let $\boldsymbol{u} \in (i\mathbb{R})^3$ have distinct coordinates, let $c \in C_c^\infty((0,1)^3)$ be real with $a_1+a_2<1$ on $\operatorname{supp} c$, and define Λ as in Lemma 64. Then $\Lambda = \int_\Omega h_{e^{-\boldsymbol{u}\cdot a}}h_c$.*

Proof. For $\Re\beta_3 > 0$ put $\boldsymbol{\sigma} = (u_1, u_2, u_3, \beta_3)$; its coordinates are distinct. For $l \le 4$ and $x \ge 0$ let $F_l(x) = \int e^{-\sum_{l' \ne l} \sigma_{l'} t_{l'}} \, \mathrm{d}t$, over $t \in [0,\infty)^{\{1,2,3,4\}\setminus\{l\}}$ with $\sum t_{l'} = x$ (two-dimensional measure), and

$$S(\boldsymbol{\sigma}; x_1, x_2) = \sum_{l=1}^{4} \frac{e^{-\sigma_l} F_l(x_1) F_l(x_2)}{\prod_{l' \ne l} (\sigma_{l'} - \sigma_l)},$$

$$\mathcal{L}_3 c(x; \beta_3) = \int c(x_1, x_2, a_3) e^{\beta_3 (1 - a_3)} \, \mathrm{d}a_3.$$

Since $\Re\sigma_l \ge 0$, $|F_l(x)| \le x^2/2$; and on $\Re\beta_3 = 2r_0$ each $|\sigma_{l'} - \sigma_l|$ is bounded below by $\min(2r_0, \min_{k \ne k'} |u_k - u_{k'}|)$, so $|S(\boldsymbol{\sigma}; x)| \ll (1 + |x|)^4$ uniformly in β_3. Also $|\mathcal{L}_3 c(x; \beta_3)| \ll_N (1 + |\beta_3|)^{-N}$ uniformly in x, and $\mathcal{L}_3 c(\cdot; \beta_3)$ vanishes off a compact subset of $(0,1)^2$. We show that both sides equal

$$\Lambda' = \int_{\mathbb{R}^2} \frac{1}{2\pi i} \int_{(2r_0)} \mathcal{L}_3 c(x; \beta_3) S(\boldsymbol{\sigma}; x) \, \mathrm{d}\beta_3 \, \mathrm{d}x.$$

(a) $\Lambda = \Lambda'$. In the ν-integral the poles inside $|\nu| = r_0$ are $\nu = -u_k$, $k \le 3$, while $\nu = -\beta_3$ lies outside as $|\beta_3| \ge 2r_0$. The residue at $\nu = -\sigma_l$ is

$$R_l(\beta) = \frac{e^{-\sigma_l}}{\prod_{l' \ne l} (\sigma_{l'} - \sigma_l) \prod_{i=1}^{2} \prod_{l' \ne l} (\beta_i + \sigma_{l'})},$$

so $\Lambda = (2\pi i)^{-3} \int \mathcal{L}c(\beta) \sum_{l \le 3} R_l(\beta) \, \mathrm{d}\beta$. We may add $l = 4$: $\mathcal{L}c(\beta) R_4(\beta) = \int c(a) e^{\beta_1 a_1 + \beta_2 a_2 - \beta_3 a_3} \, \mathrm{d}a \cdot \prod_k (u_k - \beta_3)^{-1} \prod_{i,k} (\beta_i + u_k)^{-1}$ is holomorphic in $\Re\beta_3 > 0$ and, integrating by parts in a_3, is $\ll_N e^{-a_* \Re\beta_3} (1 + |\beta_3|)^{-N}$ there, uniformly in $\Re\beta_3 \ge 2r_0$, where $a_* > 0$ is the least value of a_3 on $\operatorname{supp} c$. Moving the β_3-line to $\Re\beta_3 = R \to \infty$ shows that its β_3-integral vanishes. Next, $\int_0^\infty F_l(x) e^{-\beta_i x} \, \mathrm{d}x = \prod_{l' \ne l} (\beta_i + \sigma_{l'})^{-1}$ for $\Re\beta_i > 0$, so $\sum_l R_l(\beta) = \iint_{[0,\infty)^2} S(\boldsymbol{\sigma}; x) e^{-\beta_1 x_1 - \beta_2 x_2} \, \mathrm{d}x$. By the bounds above Fubini's theorem applies, and Laplace inversion in (β_1, β_2), applied to $\mathcal{L}c(\beta) = \iint \mathcal{L}_3 c(x; \beta_3) e^{\beta_1 x_1 + \beta_2 x_2} \, \mathrm{d}x$, gives $\Lambda = \Lambda'$.

(b) $\int_\Omega h_{e^{-\boldsymbol{u}\cdot a}} h_c = \Lambda'$. Let $F(a) = e^{-\boldsymbol{u}\cdot a}$ on $[0,1]^3$. By Lemma 6(iii), in its complex form, the left side is $\lim N^{-9} \langle E_N[F], E_N[c] \rangle_N$. Here $E_N[F] = \prod_{k=1}^{3} \det(I + X_k U)$ with $X_k = e^{-u_k/N}$. By (S5), $\overline{e_j} = \overline{D} e_{N-j}$, so $\langle E_N[F], e_A e_B e_j \rangle_N = \langle E_N[F] e_{N-j}, D e_A e_B \rangle_N$. By Laplace inversion in a_3, $c(A/N, B/N, j/N) = (2\pi i)^{-1} \int_{(2r_0)} \mathcal{L}_3 c(\frac{A}{N}, \frac{B}{N}; \beta_3) X_4^{N-j} \, \mathrm{d}\beta_3$ with $X_4 = e^{-\beta_3/N}$, and $\sum_{j=0}^{N} X_4^{N-j} e_{N-j} = \det(I + X_4 U)$. As c is real,

$$\langle E_N[F], E_N[c] \rangle_N = \sum_{A,B} \frac{1}{2\pi i} \int_{(2r_0)} \mathcal{L}_3 c\Big(\frac{A}{N}, \frac{B}{N}; \beta_3\Big) H_N(A, B; \beta_3) \, \mathrm{d}\beta_3,$$

where $H_N = \big\langle \prod_{l=1}^4 \det(I + X_l U), De_A e_B \big\rangle_N$. Only (A,B) with $A+B<N$ contribute, so Lemma 65 applies with $|X_l| \le 1$; in particular $|H_N| \le (N+1)(N+3)^6$. For fixed β_3, uniformly in A,B: $X_l^{N+3} = e^{-\sigma_l} + O(N^{-1})$, $X_l - X_{l'} = N^{-1}(\sigma_{l'} - \sigma_l)(1+O(N^{-1}))$, and $N^{-2}h_A(X_{\neq l}) = N^{-2}\sum_{m\in\mathbb{Z}^3_{\ge 0},\,|m|=A} e^{-\sum \sigma_{l'} m_{l'}/N} = F_l(A/N) + O(N^{-1})$, a Riemann sum over the simplex; so $N^{-7}H_N(A,B;\beta_3) = S(\boldsymbol{\sigma}; \frac{A}{N}, \frac{B}{N}) + o(1)$. By dominated convergence in β_3, with the majorant $\sup_x |\mathcal{L}_3 c(x;\beta_3)|$,

$$\frac{N^{-7}}{2\pi i}\int \mathcal{L}_3 c\Big(\frac{A}{N},\frac{B}{N};\beta_3\Big) H_N \,\mathrm{d}\beta_3 = G\Big(\frac{A}{N},\frac{B}{N}\Big) + o(1)$$

uniformly in (A,B), where $G(x) = \frac{1}{2\pi i}\int_{(2r_0)} \mathcal{L}_3 c(x;\beta_3) S(\boldsymbol{\sigma};x)\,\mathrm{d}\beta_3$. The function G is continuous with compact support in $(0,1)^2$, so $N^{-9}\langle E_N[F], E_N[c]\rangle_N = N^{-2}\sum_{A,B} G(\frac{A}{N},\frac{B}{N}) + o(1) \to \int G = \Lambda'$. □

The kernel identity has also been checked numerically at two points.

Proof of (42). By Lemmas 59 to 61 and 64, $\mathcal{I} = a_3 T L^9 (\int \Phi)\Lambda + o(TL^9)$, and $\Lambda = \int_\Omega h_{e^{-\mathbf{u}\cdot a}} h_c$ by Lemma 66, which applies since $\operatorname{supp} c \subset \mathcal{R}^\sharp \subset (0,1)^3$ and $a_1 + a_2 < \frac{11}{21}$ there. All bounds depend on Φ only through its support and finitely many seminorms. □

6.4 Proof of Theorem 52

Fix $\Phi \in \mathcal{W}$, and let ψ, η be real. For $\psi \in C_c^\infty(\mathcal{R})$, Lemma 53 gives $h_{\mathsf{V}\psi} = \psi$, and Lemma 31(ii) gives $\|B_\psi\|_\Phi \ll (TL^9)^{1/2}$. The coefficients of B_ψ live at $\log N/L \in I_\psi$, and those of $P_{(\mathsf{V}\psi)^J}$ at $\log N/L \in 3 - I_\psi \subset (\frac{27}{14}, 2)$. Recall that $\langle H, \mathcal{J}K\rangle_\Phi = \overline{\langle \mathcal{J}H, K\rangle_\Phi}$ for real Φ.

(a) Numerators. As in part (a) of Section 5.7, Lemma 19 and Lemma 57 give $\mathrm{N}_T(B_\psi) = \langle P_{c_0}, B_\psi\rangle_\Phi + O(T^{-A})$ with $c_0(a) = \varphi_1(|a|)\omega_0(a)$, where $\varphi_1 \in C_c^\infty((1,2))$ is 1 near I_ψ. By Proposition 58, $\mathrm{N}_T(B_\psi)/\mathcal{N}_T \to \langle h_{c_0}, \psi\rangle = \langle \varphi_1(r)v, \psi\rangle = \langle v, \psi\rangle$. Further $\mathrm{N}_T(\mathcal{J}B_\psi) = \overline{\mathrm{N}_T(B_\psi)}$ has the limit $\langle v, \psi\rangle = \langle v, \psi\circ J\rangle$.

(b) Gram entries within $\mathcal{T}_\mathcal{R}$. By Proposition 58, $\mathrm{G}_T(B_\psi, B_\eta)/\mathcal{N}_T \to \langle h_{\mathsf{V}\psi}, h_{\mathsf{V}\eta}\rangle = \langle \psi, \eta\rangle$, and $\mathrm{G}_T(\mathcal{J}B_\psi, \mathcal{J}B_\eta) = \overline{\mathrm{G}_T(B_\psi, B_\eta)}$ has the limit $\langle \psi, \eta\rangle = \langle \psi\circ J, \eta\circ J\rangle$. By Lemma 48 and the norm bound, $\mathrm{G}_T(\mathcal{J}B_\eta, B_\psi) = \langle P_{(\mathsf{V}\eta)^J}, B_\psi\rangle_\Phi + o(TL^9)$, and this is $O(T^{-A})$ by Lemma 57, as $3 - I_\eta$ and I_ψ are disjoint. This agrees with $\langle \eta\circ J, \psi\rangle = 0$, since $J\mathcal{R}\cap\mathcal{R} = \emptyset$. The entry $\mathrm{G}_T(B_\psi, \mathcal{J}B_\eta)$ is the complex conjugate.

(c) Cross-pairings with $\mathcal{T}_3^\infty$. As in part (c) of Section 5.7, since $\mathrm{G}_T(\Pi K, \mathcal{J}B_\psi) = \overline{\mathrm{G}_T(\Pi K, B_\psi)}$ and $h_{\Pi K}$ is J-symmetric, it suffices to

show $\mathrm{G}_T(K,B_\psi)/\mathcal{N}_T \to \langle h_{F_K},\psi\rangle$ and $\mathrm{G}_T(\mathcal{J}K,B_\psi)/\mathcal{N}_T \to \langle h_{F_K}\circ J,\psi\rangle$ for $K\in\{U,V,D_\phi\}$. For U and V this is Lemma 51 with W_ψ, I_ψ and $\psi^\flat$ replaced by B_ψ, I_ψ and ψ. Its proof applies verbatim: it uses $\|W_\psi\|_\Phi \ll (TL^9)^{1/2}$, which for B_ψ is Lemma 31(ii); Lemma 50, which for B_ψ is Lemma 57; and Proposition 38 for the pairing of W_ψ with a smooth compact source, which for B_ψ is Proposition 58; and in Step 4 it uses that $\psi^\flat$ vanishes unless $r\in I_\psi$, which holds for ψ. For $K=D_\phi$ the profiles $\phi(r)v$ and $\phi(3-r)v$ vanish on $1<r<2$. The coefficients of D_ϕ live at $\log N/L<\theta<1$, so $\mathrm{G}_T(D_\phi,B_\psi)=O(T^{-A})$ by Lemma 57; and $\mathrm{G}_T(\mathcal{J}D_\phi,B_\psi)=O(T^{-A})$ by the integration by parts in part (c) of Section 5.7, with $\rho=\max I_\psi<\frac{15}{14}$, since $\theta+\rho<3$.

(d) Cross-pairings with marked tests. Let $\eta\in C_c^\infty(\mathcal{Y})$. Every $a\in\operatorname{supp}\mathsf{V}\psi$ has $|a|\in I_\psi\subset(1,2)$, so Proposition 38 gives $\mathrm{G}_T(B_\psi,W_\eta)/\mathcal{N}_T\to\langle\psi,\eta^\flat\rangle$; $\mathrm{G}_T(W_\eta,B_\psi)$ is the conjugate, with the same real limit. Next $\mathrm{G}_T(B_\psi,\mathcal{J}W_\eta)=\overline{\langle\mathcal{J}B_\psi,W_\eta\rangle_\Phi}=\overline{\langle P_{(\mathsf{V}\psi)^J},W_\eta\rangle_\Phi}+o(TL^9)$ by Lemma 48 and $\|W_\eta\|_\Phi\ll(TL^9)^{1/2}$. Every $a\in\operatorname{supp}(\mathsf{V}\psi)^J$ has $|a|\in 3-I_\psi\subset(1,2)$, so Proposition 38 and Lemma 26 give the limit $\langle\psi\circ J,\eta^\flat\rangle=\langle\psi,\eta^\flat\circ J\rangle$. The entries $\mathrm{G}_T(\mathcal{J}B_\psi,\mathcal{J}W_\eta)=\overline{\mathrm{G}_T(B_\psi,W_\eta)}$ and $\mathrm{G}_T(\mathcal{J}B_\psi,W_\eta)=\overline{\mathrm{G}_T(B_\psi,\mathcal{J}W_\eta)}$ follow.

Pairs within $\mathcal{T}_4$ are covered by Theorem 23. All limits hold for every $\Phi\in\mathcal{W}$, so $\mathcal{T}_5$ is faithful.

The span. The real $\psi\in C_c^\infty(\mathcal{R})$ are dense in $L^2(\mathcal{R})$, and the $\psi\circ J$ are dense in $L^2(J\mathcal{R})$, so $\mathcal{H}(\mathcal{T}_5)\supseteq L^2(\mathcal{R}\cup J\mathcal{R})$. With $v\mathbf{1}_{S_3\cup\mathcal{Z}}\in\mathcal{H}(\mathcal{T}_4)$ from Theorem 23, $v\mathbf{1}_{S_3\cup\mathcal{Z}\cup\mathcal{R}\cup J\mathcal{R}}=v\mathbf{1}_{S_3\cup\mathcal{Z}}+v\mathbf{1}_{(\mathcal{R}\cup J\mathcal{R})\setminus(S_3\cup\mathcal{Z})}\in\mathcal{H}(\mathcal{T}_5)$.

The measure. By the proof of Theorem 23, $\Omega\setminus(S_3\cup\mathcal{Z})$ is, up to a null set, the disjoint union of N_0 and JN_0, which have traces in $(1,\frac32)$ and $(\frac32,2)$. Hence $\mathcal{R}\cap JN_0=\emptyset=J\mathcal{R}\cap N_0$, the complement of $S_3\cup\mathcal{Z}\cup\mathcal{R}\cup J\mathcal{R}$ is $(N_0\setminus\mathcal{R})\cup J(N_0\setminus\mathcal{R})$ up to a null set, and its mass is $2\mu(N_0)-2\mu(N_0\cap\mathcal{R})$. In the coordinates (r,k,x), which have Jacobian one, $N_0\cap\mathcal{R}$ is

$$1<r<\tfrac{15}{14},\qquad \tfrac12<k<\tfrac{18-7r}{21},\qquad x_{\min}<x<\tfrac{k}{2},$$
$$x_{\min}=\max\big(2k-r,\ r-k-\tfrac12\big),$$

because $x<y$ is $x<\frac{k}{2}$, $y<z$ is $x>2k-r$, $z-x<\frac12$ is $x>r-k-\frac12$, and the conditions $z<1$, $r<\frac32$ and $x>0$ are implied (if $r-k-\frac12\le0$ then $2k-r\ge r-1>0$). Writing $y-x=s$, one has $z-x=\frac{A+s}{2}$ and $z-y=\frac{A-s}{2}$ with $A=2r-3k$, so $9!v^2=\frac{9!}{64}s^2(A^2-s^2)^2$, and the x-integral is

$$\frac{9!}{128}\int_0^t s^2(A^2-s^2)^2\,\mathrm{d}s=\frac{9!}{128}\Big(\frac{A^4t^3}{3}-\frac{2A^2t^5}{5}+\frac{t^7}{7}\Big),\qquad t=k-2x_{\min}.$$

The two expressions for $x_{\min}$ cross at $k = \frac{4r-1}{6}$, which lies in the range of k exactly when $r < \frac{43}{42}$. Integrating the resulting piecewise polynomial exactly gives

$$\mu(N_0 \cap \mathcal{R}) = \frac{62748701}{30991570176} = \frac{62748701}{2^8 \cdot 3 \cdot 7^9}$$

(computed by iterated polynomial integration and, independently, by rational triangulation). With $\mu(N_0) = \frac{209}{1024}$ this gives C_0. By Proposition 8, $\liminf_{T\to\infty} M_6(T)/(c_3 T L^9) \geq C_0$. □

7 Factorable moduli just past half length

The balanced region $\mathcal{R}$ reaches shapes whose two shorter coordinates have sum $k = x + y$ up to $\frac{11}{21}$, but only for traces $r < \frac{15}{14}$. In this section we add a region of all traces $1 < r < \frac{3}{2}$ in which k lies just above $\frac{1}{2}$. Put

$$\mathcal{P} = \Big\{(a_1, a_2) \in (0,1)^2 : \ \tfrac{1}{2} < k < \tfrac{13}{25}, \ \ 5(2k-1) < a_1 < \tfrac{k}{2} - \tfrac{3}{2}(2k-1)\Big\},$$
$$\mathcal{F} = \{(x, y, z) \in \Omega : \ 1 < r < \tfrac{3}{2}, \ (x, y) \in \mathcal{P}\},$$
$$\mathcal{F}^\sharp = \{a \in (0,1)^3 : \ 1 < |a| < \tfrac{3}{2}, \ (a_1, a_2) \in \mathcal{P}\}.$$

where $k = a_1 + a_2$. On $\mathcal{P}$ we have $a_1 < \frac{k}{2} < a_2$, and the two bounds for a_1 meet exactly at $k = \frac{13}{25}$. In the coordinates

$$\varsigma = 2 - \frac{1}{k}, \qquad \mathfrak{b} = \frac{a_1}{k},$$

which we use for the arithmetic, $\mathcal{P}$ becomes $\mathcal{P}^* = \{0 < \varsigma < \frac{1}{13}, \ 5\varsigma < \mathfrak{b} < \frac{1}{2} - \frac{3}{2}\varsigma\}$. Thus $\mathcal{F} = \mathcal{F}^\sharp \cap \Omega$; and $\mathcal{F} \cap J\mathcal{F} = \emptyset$, since the traces on $J\mathcal{F}$ exceed $\frac{3}{2}$. For real $\psi \in C_c^\infty(\mathcal{F})$ the function $\mathsf{V}\psi$ is real and smooth with support in $\operatorname{supp}\psi \subset \mathcal{F}^\sharp$, and, with V and I_ψ as in Section 6, we put $F_\psi = P_{\mathsf{V}\psi}$.

Theorem 67 (Factorable row)**.** *Let*

$$\mathcal{T}_{\mathcal{F}} = \big\{(F_\psi, \psi), \ (\mathcal{J}F_\psi, \psi \circ J) : \ \psi \in C_c^\infty(\mathcal{F}) \text{ real}\big\}, \qquad \mathcal{T}_6 = \mathcal{T}_5 \cup \mathcal{T}_{\mathcal{F}}.$$

Then $\mathcal{T}_6$ is faithful, $v\mathbf{1}_{U_6} \in \mathcal{H}(\mathcal{T}_6)$ with $U_6 = S_3 \cup \mathcal{Z} \cup \mathcal{R} \cup J\mathcal{R} \cup \mathcal{F} \cup J\mathcal{F}$, and

$$\begin{aligned}\mu(U_6) &= C_0 + 2\mu\big((N_0 \cap \mathcal{F}) \setminus \mathcal{R}\big) \\ &= C_0 + \frac{840731426179152778718787}{529861775663281250000000000} \\ &= C_6 := \frac{330726844261573914237608599 3}{794792663494921875000000000} \\ &= 41.61171327466424317998\ldots.\end{aligned}$$

Overview. The proof follows that of Theorem 52; only the analytic input changes. In Section 6 the pairing of a witness P_c with a source was computed by grouping the two shorter factors of P_c into a modulus q and using the distribution of the *source* in progressions modulo q; this needed the trace of the witness to be below $\frac{15}{14}$. Here we instead treat the longest factor of the witness as a shifted zeta function, by Fourier inversion and the functional equation, so that every pairing reduces to the four-shift integral $I(\boldsymbol{\alpha})$ of Proposition 22, now with moduli $q = hk$ of size T^{κ} with $\frac{1}{2} < \kappa < \frac{13}{25}$. Its principal part is evaluated exactly as in Proposition 22. For the non-principal characters we apply the functional equations of the four Dirichlet L-functions. The resulting dual sum has length about $q^{1+\varsigma}$ and involves, at primes, the hyper-Kloosterman sums Kl_3; it must be shown to be smaller than its trivial bound by slightly more than q^{ς}. We do this by splitting the conductor as HK, where H collects the primes of the shorter factor h and K is squarefree, shifting the dual variables by multiples of H to remove the H-part, and completing modulo K, where correlation sums of Kl_3 have square-root cancellation [35]. The inequalities defining $\mathcal{P}$ are exactly the conditions under which one of three such estimates applies to every configuration of the four dual variables. The identification with Schur profiles then reuses Lemma 66.

7.1 The four-shift row

For $F \in C_c^{\infty}(\mathbb{R}^2)$, possibly complex-valued, put

$$A_F(s) = \sum_{h,k \geq 1} F\Big(\frac{\log h}{L}, \frac{\log k}{L}\Big)(hk)^{-s},$$

and let $S(\boldsymbol{\sigma}; x)$, $x \in \mathbb{R}^2$, be the kernel of Proposition 22 (equivalently of Lemma 66).

Proposition 68 (Four-shift row)**.** *Let $F \in C_c^{\infty}(\mathbb{R}^2)$ have support in a compact subset of $\mathcal{P}$, and let $\boldsymbol{\sigma} \in (i\mathbb{R})^4$. For $\Phi \in \mathcal{W}$,*

$$\begin{aligned} I_F(\boldsymbol{\sigma}) &:= \int \Phi(t/T)\, \chi(1-s) \prod_{i=1}^{4} \zeta\Big(s + \frac{\sigma_i}{L}\Big)\, \overline{A_F(s)}\, \mathrm{d}t \\ &= a_3 T L^9 \Big(\int \Phi \Big) \int_{\mathbb{R}^2} \overline{F(x)}\, S(\boldsymbol{\sigma}; x)\, \mathrm{d}x + o(TL^9). \end{aligned}$$

Proof. We follow the proof of Proposition 22 from the paragraph *Stationary phase* on, with $f_h g_k$ replaced by $\overline{F(\log h/L, \log k/L)}$ and $\varkappa$ by $\kappa_+ = \max\{a_1 +$

$a_2 : a \in \operatorname{supp} F\} < \frac{13}{25}$. The stationary-phase formula (15) uses only $\varkappa < 1$, so it holds with error $O(T^{\kappa_+ + \epsilon})$. The character expansion is unchanged, and we treat its three parts as follows.

Non-principal characters. Their total contribution is $O(T^{1-\eta})$ for some $\eta > 0$, by Proposition 76 below. This replaces the use of (16), which suffices only for $\varkappa < \frac{1}{2}$.

Principal characters off the poles. For the pair (d, ξ_0), with ξ_0 principal modulo $q = N/d$, the contour shift to $\Re w = \frac{1}{2} + \delta$ crosses only the poles at $w = 1 - \alpha_i$. On the new line the Dirichlet series is $d^{-w} \prod_i \zeta(w + \alpha_i)$ times local factors at the primes dividing N, of total size $O_\epsilon(N^\epsilon)$, and $|\tau(\xi_0)| = |\mu(q)| \le 1$. So the new line contributes $\ll_\epsilon (NT)^{\frac{1}{2}+\delta} N^\epsilon \sum_{d \mid N} d^{-\frac{1}{2}-\delta} \varphi(N/d)^{-1} \ll T^{\frac{1}{2}+\delta} N^{2\epsilon}$ for fixed N, and $O(T^{\frac{1}{2}+2\delta})$ after summing over h, k with the weight $1/N$. This replaces the bound used in Proposition 22 for these terms, which was adequate only for $\varkappa < \frac{1}{2}$.

Residues. The residue computation of Proposition 22 does not use $\varkappa < \frac{1}{2}$. In it, replace the Mellin transforms of f and g by the two-variable Laplace transform of $\overline{F}$, which decays rapidly on the lines $\Re u = \Re v' = c/L$. The local factors are unchanged, and the limit is $\int \Phi \iint \overline{F(x)} S(\boldsymbol{\sigma}; x)\, \mathrm{d}x$, first for distinct σ_i and then for all $\boldsymbol{\sigma}$ by holomorphy, as there. □

7.2 The dual sum

Fix F and $\boldsymbol{\sigma}$ as in Proposition 68, and put $\boldsymbol{\alpha} = \boldsymbol{\sigma}/L$. Let $E \subset \mathcal{P}$ be a compact set containing $\operatorname{supp} F$. For a modulus $q \ge 1$ put $X = qT/(2\pi)$ and

$$\mathcal{S}(q) = \sum_{m \ge 1} d_{\boldsymbol{\alpha}}(m)\, e(-m/q)\, \Phi(m/X),$$

which is the inner sum in (15). For $g \mid q$ put $q' = q/g$. As in Proposition 22,

$$\mathcal{S}(q) = \sum_{g \mid q} \frac{1}{\varphi(q')} \sum_{\xi \bmod q'} \tau(\bar{\xi}) \xi(-1) \sum_{n \ge 1} d_{\boldsymbol{\alpha}}(gn) \xi(n) \Phi(gn/X). \tag{44}$$

Each ξ is induced by a unique primitive character ψ modulo some $d \mid q'$, and $\tau(\bar{\xi}) = 0$ unless $r = q'/d$ is squarefree and coprime to d, in which case $\tau(\bar{\xi}) = \mu(r)\bar{\psi}(r)\tau(\bar{\psi})$ [25, Lemma 3.1]. Let $\mathcal{S}^\flat(q)$ be the part of (44) with $d > 1$. For $d \ge 1$ and $v \in \mathbb{Z}$ define

$$\mathscr{K}_d(v) = \frac{1}{d\varphi(d)} \sideset{}{^*}\sum_{\psi \bmod d} \tau(\psi)^3\, \bar{\psi}(v),$$

the sum being over primitive characters; so $\mathscr{K}_d(v) = 0$ unless $(v,d) = 1$. For $\mathfrak{a} \in \{0,1\}$ let

$$V_{\mathfrak{a}}(y) = \frac{1}{2\pi i}\int_{(-\frac{1}{10})} \widetilde{\Phi}(w)\, y^w \prod_{i=1}^{4} \frac{\Gamma\big(\frac{1-w-\alpha_i+\mathfrak{a}}{2}\big)}{\Gamma\big(\frac{w+\alpha_i+\mathfrak{a}}{2}\big)}\,\mathrm{d}w \qquad (y>0),$$

where $\widetilde{\Phi}(w) = \int_0^\infty \Phi(x) x^{w-1}\,\mathrm{d}x$.

Lemma 69 (Dual sum)**.** *Let $q \geq 1$. For each triple (g,d,r) with $gdr = q$, $d > 1$, r squarefree and $(r,d) = 1$ there are complex numbers $b_\ell = b_\ell(g,d,r)$, supported on the divisors ℓ of g^3r^4 that are coprime to d, with $\sum_\ell |b_\ell| \ll_\epsilon q^\epsilon$, such that*

$$\mathcal{S}^\flat(q) = \sum_{(g,d,r)} \mu(r)\frac{d^2\varphi(d)}{\varphi(dr)}\,\varpi_d \sum_\ell b_\ell \sum_{\mathfrak{a}\in\{0,1\}}\sum_{n\geq 1} \frac{d_{-\boldsymbol{\alpha}}(n)}{n}\, V_{\mathfrak{a}}\Big(\frac{n}{Y}\Big) \times \frac{\mathscr{K}_d(rn\bar{\ell}) + (-1)^{\mathfrak{a}}\mathscr{K}_d(-rn\bar{\ell})}{2},$$

where $\bar{\ell}$ is an inverse of ℓ modulo d, $Y = Y(g,d,\ell) = g\ell d^4/(\pi^4 X)$, $\varpi_d = \pi^{-2+\sum\alpha_i} d^{-\sum\alpha_i}$, and $d_{-\boldsymbol{\alpha}}(n) = \sum_{n_1n_2n_3n_4=n}\prod_i n_i^{\alpha_i}$. Moreover $q^{1+\varsigma} \ll Y/(\ell g^{-3}r^{-4}) \ll q^{1+\varsigma}$, where $\varsigma = 2 - \log T/\log q$.

Proof. Fix (g,d,r) and a primitive ψ modulo d, with induced character ξ modulo $q' = dr$. The Dirichlet series $D(w) = \sum_n d_{\boldsymbol{\alpha}}(gn)\xi(n)n^{-w}$ has an Euler product. At $p \nmid q$ its factor is $\prod_i(1-\psi(p)p^{-w-\alpha_i})^{-1}$. If $p \mid q'$ then $\xi(p) = 0$ and the factor is the constant $d_{\boldsymbol{\alpha}}(p^{v_p(g)})$. If $p \mid g$ and $p \nmid q'$, the factor is $\sum_{j\geq 0} d_{\boldsymbol{\alpha}}(p^{v+j})\psi(p)^j p^{-jw}$ with $v = v_p(g)$. Since $d_{\boldsymbol{\alpha}}(p^j) = h_j(p^{-\alpha_1},\dots,p^{-\alpha_4})$ is a complete homogeneous polynomial, $\prod_i(1-x_iX)\sum_{j\geq 0} h_{v+j}(x)X^j$ is a polynomial in X of degree at most 3, with coefficients bounded by $C(v+1)^3$ for $|x_i| = 1$. Hence

$$D(w) = \prod_{i=1}^{4} L(w+\alpha_i,\psi)\sum_\ell b_\ell\psi(\ell)\ell^{-w},$$

where $\sum_\ell b_\ell\psi(\ell)\ell^{-w}$ is the product over p of the local polynomials in $\psi(p)p^{-w}$ described next (so b_ℓ need not be multiplicative: for zero shifts, $g = 2$, $d = 3$ and $r = 1$ give $b_1 = 4$), b_ℓ is supported on $\ell \mid g^3r^4$ with $(\ell,d) = 1$, and it is independent of ψ: the primes $p \mid r$ contribute $d_{\boldsymbol{\alpha}}(p^{v_p(g)})\prod_i(1-\psi(p)p^{-w-\alpha_i})$, of degree 4, and the primes $p \mid g$, $p \nmid q'$ contribute the polynomial just described. The divisor bound gives $\sum|b_\ell| \ll_\epsilon q^\epsilon$.

By Mellin inversion

$$\sum_n d_{\boldsymbol{\alpha}}(gn)\xi(n)\Phi(gn/X) = \frac{1}{2\pi i}\int_{(2)} \widetilde{\Phi}(w)(X/g)^w D(w)\,\mathrm{d}w.$$

As ψ is non-principal, D is entire; $\widetilde{\Phi}$ decays rapidly on vertical lines and D grows polynomially, so we may move the line to $\Re w = -\frac{1}{10}$. There we apply the functional equation [25, Theorem 4.15]

$$L(w,\psi) = \frac{\tau(\psi)}{i^{\mathfrak{a}}\sqrt{d}}\Big(\frac{d}{\pi}\Big)^{\frac12-w}\frac{\Gamma\big(\frac{1-w+\mathfrak{a}}{2}\big)}{\Gamma\big(\frac{w+\mathfrak{a}}{2}\big)}L(1-w,\bar\psi), \qquad \psi(-1)=(-1)^{\mathfrak{a}},$$

to each factor and expand $\prod_i L(1-w-\alpha_i,\bar\psi) = \sum_n d_{-\boldsymbol{\alpha}}(n)\bar\psi(n)n^{w-1}$, which converges absolutely on this line. The ψ-dependent factors are $\tau(\bar\psi)\xi(-1)\cdot\bar\psi(r)\cdot\tau(\psi)^4 d^{-2}\cdot\psi(\ell)\bar\psi(n)$, and since $\tau(\psi)\tau(\bar\psi)=\psi(-1)d$ this equals $d^{-1}\tau(\psi)^3\bar\psi(rn\bar\ell)$. Summing over the primitive ψ of parity $\mathfrak{a}$, by means of the projector $\frac12(1+(-1)^{\mathfrak{a}}\psi(-1))$, gives $\frac{\varphi(d)}{2}(\mathscr{K}_d(v)+(-1)^{\mathfrak{a}}\mathscr{K}_d(-v))$ with $v = rn\bar\ell$. The powers combine as $(X/(g\ell))^w(d/\pi)^{2-4w-\sum\alpha_i}n^w = d^2\varpi_d\,(n/Y)^w$. Collecting the factor $\mu(r)/\varphi(q')$ and interchanging the absolutely convergent sum and integral gives the formula. Finally $X \asymp qT$ and $1\le\ell\le g^3r^4$, so $Y\ell^{-1}g^3r^4 = g^4r^4d^4/(\pi^4X) = q^4/(\pi^4X) \asymp q^3/T = q^{1+\varsigma}$. □

Lemma 70 (The dual weight). *For $j\ge0$ and $A\ge0$, uniformly for $\boldsymbol{\alpha}$ as above, $(y\partial_y)^j V_{\mathfrak{a}}(y) \ll_{j,A} \min(y^{9/10}, y^{-A})$.*

Proof. The gamma quotient $\Gamma(\frac{1-w+\mathfrak{a}}{2})/\Gamma(\frac{w+\mathfrak{a}}{2})$ is holomorphic in $\Re w<1$, and by Stirling's formula it is $\ll (1+|\Im w|)^{\frac12-\Re w}$ uniformly in vertical strips there, and $\asymp (1+|\Im w|)^{\frac12-\Re w}$ once $|\Im w|$ is large (at bounded height it can vanish, for instance at $w=0$ when $\mathfrak{a}=0$). Only the upper bound is used. As $|\Re\alpha_i|$ is 0, the integrand of $V_{\mathfrak{a}}$ is holomorphic in $\Re w<\frac{19}{20}$ and, because $\widetilde{\Phi}$ decays rapidly, may be integrated on any line $\Re w = c < \frac{19}{20}$. Applying $(y\partial_y)^j$ multiplies the integrand by w^j. Take $c=\frac{9}{10}$ for $y\le1$ and $c=-A$ for $y>1$. □

7.3 The kernel

For a prime p and $c\in\mathbb{F}_p^\times$ let $\mathrm{Kl}_3(c;p) = p^{-1}\sum_{x_1x_2x_3=c} e_p(x_1+x_2+x_3)$, the sum over $x_i\in\mathbb{F}_p^\times$, with $e_p(x)=e(x/p)$. By Deligne's theorem $|\mathrm{Kl}_3(c;p)|\le3$: the Kloosterman sheaf has rank 3 and is pure of weight 0 [33, 4.1.1].

Lemma 71 (Local structure of $\mathscr{K}_d$). (i) *If $d=\prod_t t$ with pairwise coprime prime powers t, then $\mathscr{K}_d(v) = \prod_t \mathscr{K}_t(v\,\overline{(d/t)}^3)$, the inverse taken modulo t.*

(ii) *For a prime p and $(v,p)=1$, $\mathscr{K}_p(v) = \mathrm{Kl}_3(v;p) + \frac{1}{p(p-1)}$.*

(iii) *For a prime power $t = p^a$ with $a \geq 2$, $|\mathscr{K}_t(v)| \leq 9$.*

(iv) *For a prime p, $c \in \mathbb{F}_p^\times$ and $j \in \mathbb{F}_p$, $\sum_{x\in\mathbb{F}_p} \mathscr{K}_p(cx)e_p(jx)$ vanishes if $j=0$ and has absolute value at most $2\sqrt{p}+1$ otherwise.*

In particular $|\mathscr{K}_d(v)| \leq 9^{\omega(d)} \ll_\epsilon d^\epsilon$.

Proof. (i) A character modulo d is a product $\prod_t \psi_t$ of characters modulo the t, and it is primitive if and only if every ψ_t is. By the Chinese remainder theorem $\tau(\psi) = \prod_t \psi_t(d/t)\tau(\psi_t)$, so $\tau(\psi)^3\bar{\psi}(v) = \prod_t \tau(\psi_t)^3\bar{\psi}_t(v\,\overline{(d/t)}^3)$, and $d\varphi(d) = \prod_t t\varphi(t)$.

(ii) and (iii). Let $t = p^a$ and $(v,p)=1$. Orthogonality of all characters modulo t gives

$$\frac{1}{\varphi(t)} \sum_{\psi \bmod t} \tau(\psi)^3\bar{\psi}(v) = \sum_{\substack{x_1,x_2,x_3\in(\mathbb{Z}/t\mathbb{Z})^\times \\ x_1x_2x_3=v}} e_t(x_1+x_2+x_3) =: t\,S_t(v).$$

For $a=1$ the only imprimitive character is the principal one, whose Gauss sum is -1; this gives (ii). For $a \geq 2$ every imprimitive ψ is periodic modulo p^{a-1}, and then $\tau(\psi)=0$, as one sees by summing over the p lifts of each residue modulo p^{a-1}. Hence $\mathscr{K}_t(v) = S_t(v) = t^{-1}\sum_{x,y} e_t(f(x,y))$ with $f(x,y) = x+y+v\,\overline{xy}$, the sum over units x,y modulo t.

Let $a=2m$. Write $x = x_0 + p^m u$, $y = y_0 + p^m\nu$ with x_0, y_0 running over units modulo p^m and u,ν modulo p^m. Since $\overline{x_0+p^mu} \equiv \bar{x}_0 - p^m u\bar{x}_0^2 \pmod{p^{2m}}$, we have $f(x,y) \equiv f(x_0,y_0) + p^m(f_x u + f_y\nu) \pmod{p^{2m}}$, where $f_x = 1 - v\bar{x}_0^2\bar{y}_0$ and $f_y = 1 - v\bar{x}_0\bar{y}_0^2$. Summing over u,ν gives p^{2m} if $f_x \equiv f_y \equiv 0 \pmod{p^m}$ and 0 otherwise. These congruences give $x_0 \equiv y_0 =: z$ and $z^3 \equiv v \pmod{p^m}$. The cubing map on $(\mathbb{Z}/p^m\mathbb{Z})^\times$ has fibres of size at most 3, the group being cyclic for odd p and of 2-power order for $p=2$. So $|S_t(v)| \leq 3$.

Let $a = 2m+1$, $m \geq 1$. Summing first over translations by multiples of p^{m+1}, as above, shows that only pairs $(x,y) \equiv (z,z) \pmod{p^m}$ with $z^3 \equiv v \pmod{p^m}$ contribute. For each of the at most three such classes write $x = z + p^m u$, $y = z + p^m\nu$ with u,ν modulo p^{m+1}. Expanding the inverses to second order,

$$f(x,y) \equiv f(z,z) + p^m(1-v\bar{z}^3)(u+\nu) + p^{2m}v\bar{z}^4(u^2+u\nu+\nu^2) \pmod{p^{2m+1}}.$$

As $p^m \mid 1 - v\bar{z}^3$, the phase depends only on u,ν modulo p, so the class contributes a unimodular factor times $p^{-1}\sum_{u,\nu\in\mathbb{F}_p} e_p(A(u+\nu) + B(u^2 +$

$u\nu+\nu^2)$) with $B\neq 0$. For $p\geq 5$ the quadratic form is non-degenerate and this sum has absolute value 1; for $p\leq 3$ it is at most $p\leq 3$. Hence $|S_t(v)|\leq 9$.

(iv) Extend $\psi\mapsto\sum_x\bar{\psi}(x)e_p(jx)$: it is 0 for $j=0$ and ψ non-principal, and $\psi(j)\tau(\bar{\psi})$ for $j\neq 0$. Hence the sum vanishes for $j=0$, and for $j\neq 0$, using $\tau(\psi)\tau(\bar{\psi})=\psi(-1)p$,

$$\sum_x \mathscr{K}_p(cx)e_p(jx)=\frac{1}{\varphi(p)}\sum_{\psi\neq 1}\tau(\psi)^2\bar{\psi}(-c\bar{j})=\sum_{x_1x_2=-c\bar{j}}e_p(x_1+x_2)-\frac{1}{p-1},$$

and Weil's bound for the Kloosterman sum gives the claim. The final assertion follows from (i)–(iii). □

Lemma 72 (Correlations). *Let $p\geq 5$ be prime and $m\in\{2,4\}$. For $1\leq i\leq m$ let $A_i(x)=a_i(x+b_i)$ with $a_i\in\mathbb{F}_p^\times$, $b_i\in\mathbb{F}_p$, and $\epsilon_i\in\{0,1\}$. Suppose that for all $i\neq j$ either $b_i\neq b_j$ or $a_i\neq\pm a_j$. Then for every $h\in\mathbb{F}_p$*

$$\Big|\sum_{x\in\mathbb{F}_p}e_p(hx)\prod_{i=1}^{m}\mathscr{K}_p\big(A_i(x)\big)^{[\epsilon_i]}\Big|\leq C\sqrt{p},$$

where $z^{[0]}=z$, $z^{[1]}=\bar{z}$, and C is an absolute constant.

Proof. By Lemma 71(ii), $\mathscr{K}_p(A_i(x))=\mathrm{Kl}_3(A_i(x);p)+\frac{1}{p(p-1)}$ for $x\neq -b_i$, and 0 for $x=-b_i$. Expanding the product, the terms containing a constant $\frac{1}{p(p-1)}$ and the at most m excluded points contribute $O(1)$. Since $\overline{\mathrm{Kl}_3(y;p)}=\mathrm{Kl}_3(-y;p)$, the remaining sum is $\sum_x e_p(hx)\prod_i\mathrm{Kl}_3(B_i(x);p)$ over $x\notin\{-b_i\}$, with $B_i=(-1)^{\epsilon_i}A_i$; by hypothesis $B_i\neq\pm B_j$ as affine maps for $i\neq j$. This is [35, Corollary 3.4] for the Kloosterman sheaf $\mathcal{K}\ell_3$, whose special involution is $x\mapsto -x$; that corollary includes the frequency $h=0$. We recall the argument. The function $\mathrm{Kl}_3(\cdot;p)$ is the trace function of a lisse sheaf $\mathcal{K}\ell_3$ on $\mathbf{G}_m$, of rank 3 and pure of weight 0, tamely ramified at 0 and wildly ramified at ∞ with Swan conductor 1 [33, 4.1.1], with unipotent local monodromy at 0 [33, 7.4.3], with dual $[x\mapsto -x]^*\mathcal{K}\ell_3$ [33, 4.1.3–4.1.4(2)], and with geometric monodromy group SL_3 for $p\geq 5$ [33, 11.1]. For $B\neq\pm B'$ the sheaves $[B]^*\mathcal{K}\ell_3$ and $[B']^*\mathcal{K}\ell_3$ are not geometrically isomorphic, nor is one isomorphic to the dual of the other, after any twist by a rank-one sheaf [35, §3.3, Proposition 3.7(2)–(3)]: if the roots $-b$ differ, their unipotent ramification occurs at different points, and if the roots agree, $B'=\lambda B$ with $\lambda\neq\pm 1$, and the local monodromy at ∞ distinguishes them. By the Goursat–Kolchin–Ribet criterion [34, 1.8.2] the geometric monodromy group of $\bigoplus_i[B_i]^*\mathcal{K}\ell_3$ contains SL_3^m, so $\bigotimes_i[B_i]^*\mathcal{K}\ell_3$ is geometrically irreducible

of rank $3^m > 1$ and its twist by the Artin–Schreier sheaf $\mathcal{L}_{e_p(hx)}$ has no geometric coinvariants. Deligne's theorem [32] and the Grothendieck–Ogg–Shafarevich formula then bound the sum by $C\sqrt{p}$, with C depending only on m and the ranks and conductors involved. □

7.4 Three estimates

Let $q = hk$ be a modulus, $d > 1$ a divisor of q, and c a unit modulo d. Write $d = HK$, where H is the product of the prime powers $p^a \| d$ with $a \geq 2$, or $p \mid h$, or $p \leq 3$, and K is the product of the remaining primes; so K is squarefree, $(H, K) = 1$, and every prime of K is at least 5 and divides k. By Lemma 71(i), $\mathscr{K}_d(cv) = F_H(v)G_K(v)$, where F_H is periodic modulo H with $|F_H| \leq 9^{\omega(H)}$, and $G_K(v) = \prod_{p \mid K} \mathscr{K}_p(c_p v)$ for units c_p modulo p. Throughout, W, W_1, W_2 denote smooth functions supported in $[\frac{1}{2}, 2]$ with $\|(y\partial_y)^j W\|_\infty \leq q^{j\epsilon}$ for $j \leq C_0$, a large constant, and implied constants are uniform in them; all estimates are up to factors $q^{O(\epsilon)}$, which we write $q^{o(1)}$.

Lemma 73 (One smooth variable). *For $M \geq 1$,*

$$\frac{1}{M}\Big|\sum_m W(m/M)\mathscr{K}_d(cm)\Big| \ll q^{o(1)}\Big(\frac{H\sqrt{K}}{M} + \frac{1}{\sqrt{K}}\Big).$$

Proof. By Poisson summation modulo d,

$$\sum_m W(m/M)\mathscr{K}_d(cm) = \frac{M}{d}\sum_{j \in \mathbb{Z}} \widehat{W}(jM/d)\widehat{\mathscr{K}}(j),$$

where $\widehat{W}(\xi) = \int W(y)e(-y\xi)\,\mathrm{d}y$ and $\widehat{\mathscr{K}}(j) = \sum_{x \bmod d} \mathscr{K}_d(cx)e_d(jx)$. By the Chinese remainder theorem $\widehat{\mathscr{K}}(j)$ factors over the prime powers of d; the factor at $t \mid H$ is at most $9t$, and the factor at $p \mid K$ is at most $3\sqrt{p}$ and vanishes when $p \mid j$, by Lemma 71(iv). So $\widehat{\mathscr{K}}(0) = 0$ if $K > 1$, and $|\widehat{\mathscr{K}}(j)| \leq q^{o(1)}H\sqrt{K}$ always. As $\frac{M}{d}\sum_{j \neq 0} |\widehat{W}(jM/d)| \ll q^{o(1)}(1 + M/d)$, the sum is $\ll q^{o(1)}H\sqrt{K}(1 + M/d)$; and $H\sqrt{K} \cdot M/d = M/\sqrt{K}$. If $K = 1$ the claim is trivial. □

Lemma 74 (Bilinear sums). *Let $M, N \geq 1$, and let $|\alpha_m|, |\beta_n| \leq q^{o(1)}$ be supported on $m \in [M, 2^8M]$, $n \in [N, 2^8N]$. Then*

$$\Big(\frac{1}{MN}\Big|\sum_{m,n} \alpha_m\beta_n\mathscr{K}_d(cmn)\Big|\Big)^2 \ll q^{o(1)}\Big(\frac{1}{N} + \Big(\frac{H}{M}\Big)^{1/2} + \Big(\frac{\sqrt{K}}{M}\Big)^{1/2} + K^{-1/4}\Big).$$

Proof. The left side is at most $q^{o(1)}$ by Lemma 71, so we may assume $M \geq 4H$. Let $W \geq 0$ be smooth, supported in $[\frac{1}{2}, 2^9]$ and equal to 1 on $[1, 2^8]$. By Cauchy's inequality the square of the sum is at most $q^{o(1)} M \sum_{n_1,n_2} |\beta_{n_1}\beta_{n_2}||\mathcal{C}(n_1,n_2)|$, where $\mathcal{C}(n_1,n_2) = \sum_m W(m/M)\mathscr{K}_d(cmn_1)\overline{\mathscr{K}_d(cmn_2)}$. The terms $n_1 = n_2$ give $q^{o(1)}M^2N$, that is, the term $1/N$.

Let $n_1 \neq n_2$. Write $\mathscr{K}_d(cmn_1)\overline{\mathscr{K}_d(cmn_2)} = \Psi(m)\Gamma(m)$ with $\Psi(m) = F_H(mn_1)\overline{F_H(mn_2)}$, periodic modulo H with $|\Psi| \leq q^{o(1)}$, and $\Gamma(m) = G_K(mn_1)\overline{G_K(mn_2)}$, periodic modulo K. Put $R = \lfloor M/(4H) \rfloor \geq 1$. Replacing m by $m + Hi$ does not change Ψ, so

$$\mathcal{C}(n_1,n_2) = \frac{1}{R}\sum_m \Psi(m) \sum_{i=1}^{R} W\Big(\frac{m+Hi}{M}\Big)\Gamma(m+Hi),$$

with m in an interval of length $O(M)$. By Cauchy's inequality,

$$|\mathcal{C}|^2 \leq \frac{q^{o(1)}M}{R^2} \sum_{i_1,i_2 \leq R} \Big|\sum_m W_{i_1}(m)W_{i_2}(m)\Gamma(m+Hi_1)\overline{\Gamma(m+Hi_2)}\Big|,$$

where $W_i(m) = W((m+Hi)/M)$. The terms $i_1 = i_2$ contribute $q^{o(1)}M^2/R \ll q^{o(1)}M^2H/M$. For $i_1 \neq i_2$, the inner sum is a smooth sum of a function periodic modulo K, and completing it costs a factor $\ll (M/K + \log K)$ times the largest complete sum $\sum_{x \bmod K} e_K(jx)\Gamma(x+Hi_1)\overline{\Gamma(x+Hi_2)}$. This factors over $p \mid K$. The local factor is a correlation of four functions $\mathscr{K}_p(A(x))$, $A \in \{c_pn_1(x+Hi_1),\ c_pn_2(x+Hi_1),\ c_pn_1(x+Hi_2),\ c_pn_2(x+Hi_2)\}$, two of them conjugated; it vanishes unless $p \nmid n_1n_2$. If moreover $p \nmid (i_1-i_2)(n_1^2-n_2^2)$, the hypotheses of Lemma 72 hold (the roots $-Hi_1, -Hi_2$ are distinct, and at each root the slopes c_pn_1, c_pn_2 satisfy $n_1 \neq \pm n_2$), so the local factor is $O(\sqrt{p})$; otherwise it is at most $p \cdot 4^4$. Hence the complete sum is $\ll q^{o(1)}\sqrt{K}\,(K, (i_1-i_2)(n_1^2-n_2^2))^{1/2}$, and

$$\frac{|\mathcal{C}|^2}{M^2} \ll q^{o(1)}\Big(\frac{H}{M} + \Big(\frac{\sqrt{K}}{M} + \frac{1}{\sqrt{K}}\Big)(K, n_1^2-n_2^2)^{1/2}\frac{1}{R^2}\sum_{i_1 \neq i_2}(K, i_1-i_2)^{1/2}\Big).$$

Since $\sum_{1 \leq i \leq R}(K,i)^{1/2} \leq \sum_{v \mid K} v^{1/2}R/v \leq q^{o(1)}R$, the last average is $q^{o(1)}$. Taking square roots, $|\mathcal{C}|/M \ll q^{o(1)}((H/M)^{1/2} + (\sqrt{K}/M + 1/\sqrt{K})^{1/2}(K, n_1^2 - n_2^2)^{1/4})$. Finally, with $a = n_1 - n_2 \neq 0$ and $b = n_1 + n_2$, we have $(K, ab)^{1/4} \leq (K,a)^{1/2} + (K,b)^{1/2}$, and $\sum_{n_1 \neq n_2}(K, n_1 \pm n_2)^{1/2} \leq q^{o(1)}N^2$ by the same divisor sum. Inserting these bounds proves the lemma. $\square$

Lemma 75 (Two smooth variables)**.** *Let $M \ge N \ge 1$. Then*

$$\Big(\frac{1}{MN}\Big|\sum_{m,n} W_1(m/M)W_2(n/N)\mathscr{K}_d(cmn)\Big|\Big)^2 \ll q^{o(1)}\Big(\frac{H^2}{MN}+\frac{\sqrt{K}}{M}+\frac{1}{\sqrt{K}}\Big).$$

Proof. Let Σ be the sum. If $M < 4H$ then $H^2/(MN) > \frac{1}{16}$, as $N \le M$, and the claim follows from the trivial bound; so let $M \ge 4H$. Put $I = \lfloor M/(4H)\rfloor \ge 1$ and $J = \max(1, \lfloor N/(4H)\rfloor)$. Replacing (m,n) by $(m+Hi, n+Hj)$ does not change $F_H(cmn)$, so

$$\Sigma = \frac{1}{IJ}\sum_{m,n} F_H(mn) \sum_{i<I,\, j<J} W_1\Big(\frac{m+Hi}{M}\Big)W_2\Big(\frac{n+Hj}{N}\Big) \times G_K\big((m+Hi)(n+Hj)\big),$$

where (m,n) runs over a set of $O(MN)$ points. By Cauchy's inequality $|\Sigma|^2 \le q^{o(1)}MN(IJ)^{-2}\sum_{i_1,j_1,i_2,j_2}|\Sigma_{i_1j_1i_2j_2}|$, where $\Sigma_{i_1j_1i_2j_2}$ is the sum over (m,n) of the product of the two weighted terms. The IJ diagonal terms give $|\Sigma|^2/(MN)^2 \ll q^{o(1)}/(IJ) \ll q^{o(1)}H^2/(MN)$; indeed $IJ \gg MN/H^2$ if $N \ge 8H$, and $IJ = I \gg M/H \ge MN/(8H^2)$ otherwise.

If $i_1 \ne i_2$, fix n and sum over m. At $p \mid K$ the local factor vanishes identically if $p \mid (n+Hj_1)(n+Hj_2)$. Otherwise it is the correlation of $\mathscr{K}_p(c_p(n+Hj_1)(x+Hi_1))$ with the conjugate of $\mathscr{K}_p(c_p(n+Hj_2)(x+Hi_2))$, whose roots are distinct when $p \nmid i_1-i_2$; then Lemma 72 with $m=2$ gives $O(\sqrt{p})$ for every additive frequency, and otherwise we use the bound $16p$. Completing the m-sum as in Lemma 74, $|\Sigma_{i_1j_1i_2j_2}| \ll q^{o(1)}MN(\sqrt{K}/M + 1/\sqrt{K})(K, i_1-i_2)^{1/2}$, and the average of $(K,i_1-i_2)^{1/2}$ is $q^{o(1)}$. These terms give $q^{o(1)}(\sqrt{K}/M+1/\sqrt{K})$.

If $i_1 = i_2$ and $j_1 \ne j_2$, which requires $J \ge 2$ and so $N \ge 8H$, fix m and complete in n in the same way. This gives $q^{o(1)}MN(\sqrt{K}/N + 1/\sqrt{K})(K, j_1-j_2)^{1/2}$ for each such term: at $p \mid (K, j_1-j_2)$ the two roots coincide and we use the bound $16p$. The factor $(K,j_1-j_2)^{1/2}$ averages to $q^{o(1)}$, since $\sum_{j_1\ne j_2<J}(K,j_1-j_2)^{1/2} \le \sum_{v\mid K} v^{1/2}\#\{j_1 \ne j_2 : v \mid j_1-j_2\} \le 2J^2\sum_{v\mid K} v^{-1/2} \le 2J^2\tau(K)$. These terms are a proportion $1/I \ll H/M$ of all. Since $(H/M)(\sqrt{K}/N) \le \sqrt{K}/M$, they contribute $q^{o(1)}(\sqrt{K}/M+1/\sqrt{K})$. □

7.5 Proof of the non-principal estimate

Proposition 76. *Let $E \subset \mathcal{P}$ be compact. There is $\eta > 0$ such that*

$$\sum_{\substack{h,k\geq 1\\ (\log h/L,\log k/L)\in E}} \frac{|\mathcal{S}^\flat(hk)|}{hk} \ll T^{1-\eta}.$$

Fix a pair (h,k) with $(\log h/L, \log k/L) \in E$ and put $q = hk$. All exponents below are to base q: we write $q^\kappa = T^{\kappa \log q/L}$ and so on. Put

$$\varsigma = 2 - \frac{L}{\log q}, \qquad \mathfrak{b} = \frac{\log h}{\log q}, \qquad \mathfrak{p} = \frac{\log P}{\log q},$$

where P is the largest powerful divisor of q (the product of the $p^a \| q$ with $a \geq 2$). Then $(\varsigma, \mathfrak{b})$ lies in the compact image $E^* \subset \mathcal{P}^*$ of E, and $q^{2-\varsigma} = T$. There is $c_0 \in (0, \frac{1}{100})$, depending on E, with

$$\tfrac{1}{2} - \tfrac{3}{2}\varsigma - \mathfrak{b} \geq c_0, \qquad \mathfrak{b} - 5\varsigma \geq c_0, \qquad \tfrac{1}{13} - \varsigma \geq c_0 \qquad \text{on } E^*. \tag{45}$$

Lemma 77 (Block estimate)**.** *There are $\delta > 0$ and $\epsilon_0 > 0$, depending only on E, with the following property. Let (g,d,r) and ℓ be as in Lemma 69, put $\mathfrak{g} = \log g/\log q$, $\mathfrak{r} = \log r/\log q$, let c be a unit modulo d, and let $N_1, \dots, N_4 \geq 1$ with $N_* = N_1N_2N_3N_4 \leq Yq^{\epsilon_0}$. Put $\mathfrak{v} = \max(0, \log(Y/N_*)/\log q)$ and*

$$\tau = \varsigma - 2\mathfrak{g} - 3\mathfrak{r} - \tfrac{1}{2}\mathfrak{p} - \tfrac{9}{10}\mathfrak{v}.$$

Then, for weights w_i as in the previous subsection,

$$\Sigma := \frac{1}{N_*}\Big|\sum_{n_1,\dots,n_4}\prod_{i=1}^{4} w_i(n_i/N_i)\,\mathscr{K}_d(cn_1n_2n_3n_4)\Big| \ll q^{-\tau-\delta}.$$

Proof. Write $N_i = q^{\nu_i}$, $D = \nu_1 + \cdots + \nu_4$, and order the indices so that $\nu_1 \geq \nu_2 \geq \nu_3 \geq \nu_4 \geq 0$. Let $d = HK$ be as in the previous subsection. Every prime power in H with exponent at least 2 divides P, so $H \leq 6Ph$; every prime of K divides $k = q/h$; and $d = q/(gr)$. Hence, with

$$\mathfrak{h} = \mathfrak{b} + \mathfrak{p}, \qquad \mathfrak{k} = \tfrac{1}{2}(1 - \mathfrak{b}),$$

we have $H \leq q^{\mathfrak{h}+o(1)}$, $\sqrt{K} \leq q^{\mathfrak{k}}$, $K \geq q^{1-\mathfrak{b}-\mathfrak{p}-\mathfrak{g}-\mathfrak{r}-o(1)}$ and $H\sqrt{K} = \sqrt{Hd} \leq q^{(1+\mathfrak{b}+\mathfrak{p}-\mathfrak{g}-\mathfrak{r})/2+o(1)}$. Since $\ell \geq 1$, Lemma 69 gives $Y \geq q^{1+\varsigma-3\mathfrak{g}-4\mathfrak{r}-o(1)}$, so

$$D \geq \lambda - o(1), \qquad \lambda = 1 + \varsigma - 3\mathfrak{g} - 4\mathfrak{r} - \mathfrak{v}.$$

The trivial bound is $\Sigma \le q^{o(1)}$, by Lemma 71. Let $\delta = c_0/100$ and $\tau' = \tau + 2\delta$. If $\tau' \le 0$ the trivial bound suffices, so assume $\tau' > 0$; it is enough to show $\Sigma \ll q^{-\tau'+o(1)}$. Note that $\tau' \le \varsigma + 2\delta$. By direct expansion (also checked by computer) we have the identities

$$\begin{aligned}
\lambda - 8\tau &= (1-7\varsigma) + 13\mathfrak{g} + 20\mathfrak{r} + 4\mathfrak{p} + \tfrac{31}{5}\mathfrak{v},\\
(1-\mathfrak{b}-\mathfrak{p}-\mathfrak{g}-\mathfrak{r}) - 8\tau &= (1-\mathfrak{b}-8\varsigma) + 15\mathfrak{g} + 23\mathfrak{r} + 3\mathfrak{p} + \tfrac{36}{5}\mathfrak{v},\\
\lambda - 3\tau - \tfrac{1}{2}(1+\mathfrak{b}+\mathfrak{p}-\mathfrak{g}-\mathfrak{r}) &= (\tfrac{1}{2}(1-\mathfrak{b}) - 2\varsigma) + \tfrac{7}{2}\mathfrak{g} + \tfrac{11}{2}\mathfrak{r} + \mathfrak{p} + \tfrac{17}{10}\mathfrak{v},\\
\lambda - 4\tau - 2\mathfrak{h} &= (1 - 3\varsigma - 2\mathfrak{b}) + 5\mathfrak{g} + 8\mathfrak{r} + \tfrac{13}{5}\mathfrak{v},\\
\lambda - 6\tau - 2\mathfrak{k} &= (\mathfrak{b} - 5\varsigma) + 9\mathfrak{g} + 14\mathfrak{r} + 3\mathfrak{p} + \tfrac{22}{5}\mathfrak{v}.
\end{aligned}$$

By (45) the constant terms on the right are at least c_0; for instance $1-\mathfrak{b}-8\varsigma \ge \frac{1}{2} - \frac{13}{2}\varsigma \ge c_0$. Replacing τ by τ' lowers each left side by at most $16\delta \le c_0/6$. Hence, for q large,

$$\begin{gathered}
D > 8\tau', \qquad \log K/\log q > 8\tau', \qquad \lambda - 3\tau' > \tfrac{1}{2}(1+\mathfrak{b}+\mathfrak{p}-\mathfrak{g}-\mathfrak{r}),\\
D - 4\tau' > 2\mathfrak{h}, \qquad D - 6\tau' > 2\mathfrak{k},
\end{gathered} \tag{46}$$

each with a margin of at least $c_0/2$. We now consider three cases; in each, the margins in (46) absorb the factors $q^{o(1)}$.

Case 1: a good grouping. Suppose that some set $\mathcal{I} \subset \{1,2,3,4\}$ has $\sum_{i\in\mathcal{I}} \nu_i > 2\tau'$ and $\sum_{i\notin\mathcal{I}} \nu_i > \max(\mathfrak{h},\mathfrak{k}) + 4\tau'$. Group the variables n_i, $i \notin \mathcal{I}$, into m and the others into n; the coefficients of m and n are bounded by divisor functions, and are supported in ranges $[M, 2^8M]$ and $[N, 2^8N]$ with $M \asymp \prod_{i\notin\mathcal{I}} N_i$ and $N \asymp \prod_{i\in\mathcal{I}} N_i$. In Lemma 74 the four terms are then at most $q^{-2\tau'}$, by the two assumed inequalities and $K > q^{8\tau'}$, so $\Sigma^2 \ll q^{-2\tau'+o(1)}$.

Case 2: one long variable. Suppose that $\nu_2+\nu_3+\nu_4 \le 2\tau'$. Fix n_2, n_3, n_4; then $cn_2n_3n_4$ is a unit modulo d or the inner sum vanishes. Lemma 73 with $M = N_1 \ge q^{D-2\tau'}$ gives $\Sigma \ll q^{o(1)}(H\sqrt{K}/N_1 + 1/\sqrt{K})$, and by (46) both terms are $\ll q^{-\tau'}$, since $D - 2\tau' - \tau' \ge \lambda - 3\tau' - o(1)$.

Case 3: two long variables. Suppose that $\nu_3+\nu_4 \le 2\tau'$. Fix n_3, n_4; again cn_3n_4 is a unit modulo d or the inner sum vanishes. Apply Lemma 75 with $M = N_1 \ge N = N_2$. Here $\nu_1+\nu_2 \ge D - 2\tau' > 2\mathfrak{h} + 2\tau'$ and $\nu_1 \ge \frac{1}{2}(D-2\tau') > \mathfrak{k} + 2\tau'$, by (46); so $\Sigma^2 \ll q^{-2\tau'+o(1)}$, using $K > q^{8\tau'}$ for the last term.

It remains to show that one of the three cases occurs. Put $\Lambda = \max(\mathfrak{h},\mathfrak{k}) + 4\tau'$, $\Delta = \Lambda - D/2$ and $U = D - \Lambda$.

If $\Delta < 0$, then $\nu_1 \ge D/4 > 2\tau'$. Either $\nu_2+\nu_3+\nu_4 \le 2\tau'$, which is Case 2, or $\{2,3,4\}$ and $\{1\}$ both have degree above $2\tau'$, and the larger of the two has degree at least $D/2 > \Lambda$, which is Case 1.

Let $\Delta \geq 0$. We claim that $3U > D$, $2\Delta < U$ and $4\tau' < U$. Put $\Gamma = \frac{13}{2}\mathfrak{g} + 10\mathfrak{r} + \mathfrak{p} + \frac{31}{10}\mathfrak{v} \geq 0$. By direct expansion (§2 of the same script),

$$\mathfrak{h} + 4\tau - \tfrac{\lambda}{2} = \big(\mathfrak{b} - \tfrac{1}{2} + \tfrac{7}{2}\varsigma\big) - \Gamma, \qquad \mathfrak{k} + 4\tau - \tfrac{\lambda}{2} = \big(\tfrac{7}{2}\varsigma - \tfrac{1}{2}\mathfrak{b}\big) - \Gamma - \mathfrak{p}.$$

By (45), $\mathfrak{b} - \frac{1}{2} + \frac{7}{2}\varsigma \leq 2\varsigma - c_0$ and $\frac{7}{2}\varsigma - \frac{1}{2}\mathfrak{b} \leq \varsigma - \frac{c_0}{2}$. As $\tau' = \tau + 2\delta$, $8\delta < \frac{c_0}{4}$ and $D \geq \lambda - o(1)$, we get, for q large,

$$\Lambda - \tfrac{\lambda}{2} < 2\varsigma - \Gamma - \tfrac{c_0}{4}, \qquad \text{so} \qquad 0 \leq \Delta < 2\varsigma - \Gamma - \tfrac{c_0}{8}.$$

First, $\Gamma < 2\varsigma$ gives $3\mathfrak{g}+4\mathfrak{r}+\mathfrak{v} < \varsigma$, hence $\lambda > 1$ and $D > \frac{12}{13}$; so $\Delta < 2\varsigma < \frac{2}{13} < \frac{D}{6}$, and $U = \frac{D}{2} - \Delta > \frac{D}{3}$, which gives $3U > D$ and $2\Delta < \frac{D}{3} < U$. Second, $U = D - \Lambda \geq \frac{\lambda}{2} - (\Lambda - \frac{\lambda}{2}) - o(1) > \frac{\lambda}{2} - 2\varsigma + \Gamma$, and $\Gamma \geq \frac{1}{2}(3\mathfrak{g} + 4\mathfrak{r} + \mathfrak{v}) = \frac{1}{2}(1 + \varsigma - \lambda)$, so $U > \frac{1}{2} - \frac{3}{2}\varsigma$. Since $\tau \leq \varsigma$ and $\varsigma < \frac{1}{13}$,

$$U - 4\tau' > \tfrac{1}{2} - \tfrac{11}{2}\varsigma - 8\delta > \tfrac{1}{13} - 8\delta > 0.$$

This proves the claim.

Now if some $\mathcal{I}$ has $\sum_{i\in\mathcal{I}} \nu_i \in (2\tau', U)$, then its complement has degree above $D - U = \Lambda$, which is Case 1. Otherwise every partial sum of the ν_i lies in $[0, 2\tau'] \cup [U, D]$. Since $D > 8\tau'$, some ν_i, namely ν_1, is at least U, and at most two are, as $3U > D$. If only $\nu_1 \geq U$, then $\nu_2, \nu_3, \nu_4 \leq 2\tau'$; the partial sums ν_2, $\nu_2 + \nu_3$, $\nu_2 + \nu_3 + \nu_4$ increase by at most $2\tau'$ at each step and cannot enter $(2\tau', U)$, which has length greater than $2\tau'$; so $\nu_2 + \nu_3 + \nu_4 \leq 2\tau'$, which is Case 2. If $\nu_1, \nu_2 \geq U$, then $\nu_3 + \nu_4 \leq D - 2U = 2\Delta < U$, so $\nu_3 + \nu_4 \leq 2\tau'$, which is Case 3. □

The case analysis has also been tested on random parameters: of 200,000 parameter draws, 30,575 satisfy the selection guards, and for each of these some case gives a saving larger than τ. This is a sanity check; the proof is the argument above.

Proof of Proposition 76. Fix (h, k) and $q = hk$ as above, and a term $(g, d, r, \ell, \mathfrak{a})$ of Lemma 69. Insert smooth dyadic partitions of unity $1 = \sum_{N_i} W(n_i/N_i)$ for the four variables of $d_{-\boldsymbol{\alpha}}(n) = \sum_{n_1\cdots n_4 = n} \prod_i n_i^{\alpha_i}$. For a fixed quadruple put $N_* = \prod N_i$; then $n = n_1 \cdots n_4 \in [N_*/16, 16N_*]$, and $n^{-1}V_{\mathfrak{a}}(n/Y) = \vartheta(n/N_*)$, where $\vartheta(y) = \bar{W}(y)(yN_*)^{-1}V_{\mathfrak{a}}(yN_*/Y)$ for a fixed $\bar{W} \in C_c^\infty((\frac{1}{32}, 32))$ equal to 1 on $[\frac{1}{16}, 16]$. By Mellin inversion $\vartheta(y) = \frac{1}{2\pi}\int \widehat{\vartheta}(t) y^{-it}\,\mathrm{d}t$, and integration by parts and Lemma 70 give

$$|\widehat{\vartheta}(t)| \ll_{j,A} (1 + |t|)^{-j} N_*^{-1} \min\big((N_*/Y)^{9/10}, (N_*/Y)^{-A}\big).$$

The part $|t| > q^{\epsilon}$ is negligible by the trivial bound. For $|t| \le q^{\epsilon}$ the factor $y^{-it} = \prod_i (n_i/N_i)^{-it}$ separates the variables, and the weights $w_i(y) = W(y)(yN_i)^{\alpha_i} y^{-it}$ satisfy the hypotheses of the previous subsection. If $N_* > Yq^{\epsilon_0}$ the factor $(N_*/Y)^{-A}$ makes the quadruple negligible. Otherwise Lemma 77 applies, and $\min\big((N_*/Y)^{9/10}, (N_*/Y)^{-A}\big) \le q^{-\frac{9}{10}\mathfrak{v}}$: if $N_* \le Y$ the first term equals $q^{-\frac{9}{10}\mathfrak{v}}$, and if $N_* > Y$ then $\mathfrak{v} = 0$ and the second term is less than 1. Summing over the $O(L^4)$ quadruples, the n-sum in Lemma 69 is $\ll q^{-\frac{9}{10}\mathfrak{v} - \tau - \delta + o(1)} \le q^{-\varsigma + 2\mathfrak{g} + 3\mathfrak{r} + \frac{1}{2}\mathfrak{p} - \delta + o(1)}$.

The prefactor is $d^2\varphi(d)/\varphi(dr) = d^2/\varphi(r) \le q^{2-2\mathfrak{g}-3\mathfrak{r}+o(1)}$, since $d = q/(gr)$. The number of triples (g,d,r), the sum $\sum_\ell |b_\ell|$ and $|\varpi_d|$ are $q^{o(1)}$. Hence

$$|\mathcal{S}^\flat(q)| \ll q^{2-\varsigma+\frac{1}{2}\mathfrak{p}-\delta+o(1)} = T\,P^{1/2} q^{-\delta+o(1)}.$$

Finally $q \ge T^{1/2}$, and $\sum_{q\le Z} P(q)^{1/2} q^{-1} \le \sum_{P\le Z} P^{1/2} \cdot \frac{1}{P} \sum_{m \le Z/P} \frac{1}{m} \ll (\log Z)^2$, since there are $O(\sqrt{x})$ powerful numbers up to x. Summing over (h,k), with at most $\tau(q)$ pairs for each $q \le T$, gives $\ll T^{1-\delta/2+o(1)}$. □

7.6 The factorable row

Proposition 78 (Factorable pairing). *Let $c \in C_c^\infty(\mathbb{R}^3)$ be real with $\operatorname{supp} c \subset \mathcal{F}^\sharp$, and let $b \in C_c^\infty(\mathbb{R}^3)$. Uniformly for Φ in bounded subsets of C_c^∞ with a fixed compact support in $(1,2)$,*

$$\langle P_b, P_c\rangle_\Phi = a_3 T L^9 \Big(\int \Phi\Big) \int_\Omega h_b h_c + o(TL^9).$$

Proof. The reduction to elementary sources in the proof of Proposition 58 uses only Lemma 57, Lemma 33, Lemma 31 and the fact that h_c vanishes unless $r \in I_c$; so it suffices to prove that for $\boldsymbol{u} \in (i\mathbb{R})^3$ with distinct coordinates and ϕ as there,

$$\langle P_{\boldsymbol{u},\phi}, P_c\rangle_\Phi = a_3 T L^9 \Big(\int \Phi\Big) \int_\Omega h_{e^{-\boldsymbol{u}\cdot a} h_c} + o(TL^9). \tag{47}$$

Let $Z(s) = \prod_{k=1}^3 \zeta(s + u_k/L)$. By Lemma 19, $Z(s) = \sum_n b_{\boldsymbol{u}}(n)\omega(n/T^8) n^{-s} + O(T^{-A})$, and the coefficients of this polynomial minus $P_{\boldsymbol{u},\phi}$ vanish for $\log n/L$ near I_c; so $\langle P_{\boldsymbol{u},\phi}, P_c\rangle_\Phi = \langle Z, P_c\rangle_\Phi + O(T^{-A})$ by Lemma 57.

Fourier inversion in the third coordinate. For $\beta \in i\mathbb{R}$ and $x \in \mathbb{R}^2$ let, as in Lemma 66,

$$\mathcal{L}_3 c(x;\beta) = \int c(x_1, x_2, a_3) e^{\beta(1-a_3)}\, \mathrm{d}a_3.$$

By Fourier inversion $c(x,a_3) = \frac{1}{2\pi i}\int_{(0)} \mathcal{L}_3 c(x;\beta) e^{-\beta(1-a_3)}\,\mathrm{d}\beta$; also $|\mathcal{L}_3 c(x;\beta)| \ll_j (1+|\beta|)^{-j}$; moreover the function $F_\beta = \overline{\mathcal{L}_3 c(\cdot;\beta)}$ on $\mathbb{R}^2$ is supported in the compact projection E of $\operatorname{supp} c$, which lies in $\mathcal{P}$, and its Fourier transform satisfies $\int |\widehat{F_\beta}| \ll_j (1+|\beta|)^{-j}$. On the critical line $\overline{n^{-s}} = n^{-(1-s)}$, and c is real, so

$$\overline{P_c(s)} = \frac{1}{2\pi i}\int_{(0)} e^{-\beta}\,\overline{A_{F_\beta}(s)} \sum_m \omega(m/T^8) m^{-(1-s-\beta/L)}\,\mathrm{d}\beta,$$

the m-sum being finite. By Lemma 19, whose proof is uniform for shifts of size at most $T^{1/2}$, the m-sum is $\zeta(1-s-\beta/L)+O(T^{-A})$ for $|\beta|\le T^{1/2}$, and the rest of the β-line is negligible. By the functional equation $\zeta(1-s-\beta/L) = \chi(1-s-\beta/L)\zeta(s+\beta/L)$, and by Stirling's formula

$$e^{-\beta}\chi(1-s-\beta/L) = \chi(1-s)\big(1+O((1+|\beta|)/L)\big) \qquad (T\le t\le 2T,\ |\beta|\le L).$$

Put $\boldsymbol{\sigma}(\beta) = (u_1,u_2,u_3,\beta)$. We obtain

$$\langle Z, P_c\rangle_\Phi = \frac{1}{2\pi i}\int_{(0)} \Big(I_{F_\beta}(\boldsymbol{\sigma}(\beta)) + \mathcal{E}(\beta)\Big)\,\mathrm{d}\beta + O(T^{-A}),$$

where I_F is as in Proposition 68 and, by Cauchy–Schwarz, $|\mathcal{E}(\beta)| \ll \frac{1+|\beta|}{L}\|Z\|_\Phi \|\zeta(s+\beta/L)A_{F_\beta}\|_\Phi$ for $|\beta|\le L$; for $|\beta|>L$ we use $|e^{-\beta}\chi(1-s-\beta/L)/\chi(1-s)| \ll 1$ instead.

Norms. By Hölder's inequality, Harper's bound and Lemma 31(i) (applied to the Fourier superpositions $A_{F_\beta} = (2\pi)^{-2}\int \widehat{F_\beta}(\xi) A_{g_{\xi_1}} A_{g_{\xi_2}}\,\mathrm{d}\xi$ as in Lemma 31(ii)), $\|Z\|_\Phi \ll (TL^9)^{1/2}$ and $\|\zeta(s+\beta/L)A_{F_\beta}\|_\Phi \ll_j (TL^9)^{1/2}(1+|\beta|)^{-j}$ for $|\beta|\le LT/4$; the same bounds give $|I_{F_\beta}(\boldsymbol{\sigma}(\beta))| \ll_j TL^9(1+|\beta|)^{-j}$ there, and the trivial bound with the decay of $\mathcal{L}_3 c$ handles larger $|\beta|$. Hence $\int|\mathcal{E}| = o(TL^9)$.

The limit. For each fixed $\beta\in i\mathbb{R}$, Proposition 68 gives $I_{F_\beta}(\boldsymbol{\sigma}(\beta)) = a_3 TL^9(\int\Phi)\int \mathcal{L}_3 c(x;\beta) S(\boldsymbol{\sigma}(\beta);x)\,\mathrm{d}x + o(TL^9)$. By dominated convergence (with the majorant $TL^9(1+|\beta|)^{-j}$ from the norm bounds),

$$\langle Z, P_c\rangle_\Phi = a_3 TL^9\Big(\int\Phi\Big)\frac{1}{2\pi i}\int_{(0)}\int_{\mathbb{R}^2} \mathcal{L}_3 c(x;\beta) S(\boldsymbol{\sigma}(\beta);x)\,\mathrm{d}x\,\mathrm{d}\beta + o(TL^9),$$

which makes sense also when $\int\Phi = 0$. The integrand is entire in β. For $0\le \Re\beta\le 2r_0$ the function $S(\boldsymbol{\sigma}(\beta);x)$ is bounded polynomially in $|\beta|$, uniformly for x in a compact set: away from the points $\beta = u_k$ this follows from the explicit formula for S, as in the proof of Lemma 66, and near them from the

maximum principle. Since $\mathcal{L}_3 c(x;\beta)$ decays rapidly in this strip, we may move the line to $\Re\beta = 2r_0$, and the double integral becomes Λ' in the proof of Lemma 66. Part (b) of that proof uses only that $c \in C_c^\infty((0,1)^3)$ is real with $a_1 + a_2 < 1$ on its support, which holds here, and gives $\Lambda' = \int_\Omega h_{e^{-\mathbf{u}\cdot a}} h_c$. This proves (47). $\square$

7.7 Proof of Theorem 67

Fix $\Phi \in \mathcal{W}$. For real $\psi \in C_c^\infty(\mathcal{F})$, Lemma 53 gives $h_{\mathsf{V}\psi} = \psi$, Lemma 31(ii) gives $\|F_\psi\|_\Phi \ll (TL^9)^{1/2}$, and the coefficients of F_ψ live at $\log N/L \in I_\psi \subset (1, \frac{3}{2})$. We go through the proof of Theorem 52, parts (a)–(d), with $\mathcal{R}$, B_ψ and Proposition 58 replaced by $\mathcal{F}$, F_ψ and Proposition 78. Every step applies verbatim: part (a) uses only the pairing of the witness with a compact source; part (b) uses, for the reflected entries, Lemma 48, which requires $\operatorname{supp} \mathsf{V}\psi \subset (0,1)^3$, and Lemma 57 with the disjoint trace ranges I_η and $3 - I_\psi \subset (\frac{3}{2}, 2)$; part (c) uses Lemma 51 through the same three properties, and for $\mathcal{J}D_\phi$ the bound $\theta + \max I_\psi < 3$; and part (d) uses that $I_\psi \subset (1,2)$, as Proposition 38 requires.

It remains to treat the cross-pairings between $\mathcal{T}_\mathcal{R}$ and $\mathcal{T}_\mathcal{F}$. For real $\psi \in C_c^\infty(\mathcal{F})$ and $\eta \in C_c^\infty(\mathcal{R})$, Proposition 58 with $c = \mathsf{V}\eta$ and $b = \mathsf{V}\psi$ gives $\mathrm{G}_T(F_\psi, B_\eta)/\mathcal{N}_T \to \langle \psi, \eta\rangle$. By Lemma 48, $\mathrm{G}_T(\mathcal{J}F_\psi, B_\eta) = \langle P_{(\mathsf{V}\psi)^J}, B_\eta\rangle_\Phi + o(TL^9)$, and Proposition 58 with $b = (\mathsf{V}\psi)^J$ and Lemma 26 give $\mathrm{G}_T(\mathcal{J}F_\psi, B_\eta)/\mathcal{N}_T \to \langle \psi \circ J, \eta\rangle$ (which is 0). Likewise $\mathrm{G}_T(\mathcal{J}B_\eta, F_\psi)/\mathcal{N}_T \to \langle \eta \circ J, \psi\rangle$ by Lemma 48 and Proposition 78 with $b = (\mathsf{V}\eta)^J$. The remaining entries are complex conjugates of these, by $\langle \mathcal{J}H, \mathcal{J}K\rangle_\Phi = \overline{\langle H, K\rangle_\Phi}$ and $\langle H, \mathcal{J}K\rangle_\Phi = \overline{\langle \mathcal{J}H, K\rangle_\Phi}$, and all the limits are real. Hence $\mathcal{T}_6$ is faithful.

The span. As in Theorem 52, $\mathcal{H}(\mathcal{T}_6) \supseteq L^2(\mathcal{F} \cup J\mathcal{F})$, and with $v\mathbf{1}_{S_3 \cup \mathcal{Z} \cup \mathcal{R} \cup J\mathcal{R}} \in \mathcal{H}(\mathcal{T}_5)$ we get $v\mathbf{1}_{U_6} \in \mathcal{H}(\mathcal{T}_6)$.

The measure. Since $\mathcal{F} \subset \{1 < r < \frac{3}{2}\}$, we have $\mathcal{F} \cap JN_0 = \emptyset = J\mathcal{F} \cap N_0$, and as in the proof of Theorem 52 the complement of U_6 is, up to a null set, $(N_0 \setminus (\mathcal{R} \cup \mathcal{F})) \cup J(N_0 \setminus (\mathcal{R} \cup \mathcal{F}))$. So $\mu(U_6) = C_0 + 2\mu((N_0 \cap \mathcal{F}) \setminus \mathcal{R})$. We describe this set in the coordinates (k, x, z), $k = x + y$, which have Jacobian one. On $k > \frac{1}{2}$ the inequality $21k + 7r < 18$ forces $r < \frac{15}{14}$, so outside $\mathcal{R}$ means $z \geq z_0(k) := \frac{18 - 28k}{7}$. Together with $z - x < \frac{1}{2}$ and the conditions defining $\mathcal{F}$, the set is

$$\tfrac{1}{2} < k < \tfrac{13}{25}, \qquad x_{\min}(k) < x < \tfrac{k}{2} - \tfrac{3}{2}(2k-1), \qquad z_0(k) < z < \tfrac{1}{2} + x,$$
$$x_{\min}(k) = \max\big(5(2k-1),\, z_0(k) - \tfrac{1}{2}\big).$$

The remaining conditions are implied: $y = k - x < 5 - 9k < z_0(k) < z$, as $k > \frac{17}{35}$; $z < \frac{1}{2} + x < 1$; $r > k + z_0(k) = \frac{18-21k}{7} > 1$, as $k < \frac{11}{21}$; and $r < k + \frac{1}{2} + x < \frac{3}{2}$, as $x \le \frac{k}{2}$ and $k < \frac{2}{3}$. The two lower bounds for x cross at $k = \frac{99}{196}$. Since $9!v^2 = \frac{9!}{4}(k-2x)^2(z-x)^2(z-k+x)^2$,

$$2\mu\big((N_0 \cap \mathcal{F}) \setminus \mathcal{R}\big) = \frac{9!}{2}\int_{1/2}^{13/25}\int_{x_{\min}(k)}^{\frac{k}{2}-\frac{3}{2}(2k-1)}\int_{z_0(k)}^{\frac{1}{2}+x}(k-2x)^2(z-x)^2(z-k+x)^2\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}k,$$

a sum of two polynomial integrals over rational limits, equal to $\frac{840731426179152778718787}{529861775663281250000000000}$. This value is computed by exact polynomial integration and, independently, by an exact quadrature rule; the second computation also checks the inclusions above at the vertices of the two cells. By Proposition 8, $\liminf_{T\to\infty} M_6(T)/(c_3 T L^9) \ge \mu(U_6)$. □

8 The mixed row up to $\kappa < \frac{3}{5}$

The region $\mathcal{F}$ of Section 7 is limited by Proposition 76, which bounds the non-principal part of the four-shift row modulus by modulus. In this section we keep the sum over the moduli $q = hk$ with its signs and its weights. After a double Poisson transform the dual sum of Lemma 69 becomes a sum of Kloosterman fractions, and we bound it in four ways: by the bound of Deshouillers and Iwaniec for such sums averaged over the modulus, by the same bound with the two marked factors of q kept as summation variables, by completing a reciprocal frequency with Weil's bound, and by the Kuznetsov formula at the level fixed by the denominators. Each of these bounds is an affine inequality between the degrees of the lengths involved, and the remaining problem is to show that for every configuration one of them saves a power. For the two smallest ranges of ς we do this in prose (Lemma 90(i)–(ii)), and check the prose by machine; for the three larger bands we give exact rational certificates.

For $\frac{1}{2} < k < \frac{3}{5}$ put

$$x_+(k) = \begin{cases} \frac{k}{2}, & \frac{1}{2} < k < \frac{10}{19}, \\ \max\left(\frac{21-39k}{2}, \frac{7-12k}{4}\right), & \frac{10}{19} \le k < \frac{7}{13}, \\ \frac{2-3k}{4}, & \frac{7}{13} \le k < \frac{4}{7}, \\ \frac{3-5k}{2}, & \frac{4}{7} \le k < \frac{3}{5}, \end{cases} \tag{48}$$

and

$$\mathcal{P}^{+}=\Big\{(a_1,a_2)\in(0,1)^2:\ \tfrac{1}{2}<k<\tfrac{3}{5},\ a_1<x_{+}(k)\Big\},\qquad k=a_1+a_2,$$
$$\mathcal{F}^{+}=\{(x,y,z)\in\Omega:\ 1<r<\tfrac{3}{2},\ (x,y)\in\mathcal{P}^{+}\},$$
$$\mathcal{F}^{+\sharp}=\{a\in(0,1)^3:\ 1<|a|<\tfrac{3}{2},\ (a_1,a_2)\in\mathcal{P}^{+}\}.$$

Since $x_{+}(k)\le\frac{k}{2}$, we have $a_1<a_2$ on $\mathcal{P}^{+}$; the set $\mathcal{P}^{+}$ is open, because x_{+} is continuous except at $k=\frac{10}{19}$ and $k=\frac{7}{13}$, where it jumps down; and $\mathcal{P}\subset\mathcal{P}^{+}$. In the coordinates $\varsigma=2-1/k$, $\mathfrak{b}=a_1/k$ of Section 7, $\mathcal{P}^{+}$ becomes $\mathcal{P}^{+*}=\{0<\varsigma<\frac{1}{3},\ 0<\mathfrak{b}<\mathfrak{b}_{+}(\varsigma)\}$, where

$$\mathfrak{b}_{+}(\varsigma)=\begin{cases}\frac{1}{2}, & 0<\varsigma<\frac{1}{10},\\ \max\left(\frac{3-21\varsigma}{2},\frac{2-7\varsigma}{4}\right), & \frac{1}{10}\le\varsigma<\frac{1}{7},\\ \frac{1-2\varsigma}{4}, & \frac{1}{7}\le\varsigma<\frac{1}{4},\\ \frac{1-3\varsigma}{2}, & \frac{1}{4}\le\varsigma<\frac{1}{3}.\end{cases}$$

The first entry of the maximum is the larger one exactly for $\varsigma\le\frac{4}{35}$, that is $k\le\frac{35}{66}$. As in Section 7, $\mathcal{F}^{+}=\mathcal{F}^{+\sharp}\cap\Omega$ and $\mathcal{F}^{+}\cap J\mathcal{F}^{+}=\emptyset$, and for real $\psi\in C_c^{\infty}(\mathcal{F}^{+})$ we put $F_\psi=P_{\mathsf{V}\psi}$.

Theorem 79 (Mixed row)**.** *Let*

$$\mathcal{T}_{\mathcal{F}^{+}}=\big\{(F_\psi,\psi),\ (\mathcal{J}F_\psi,\psi\circ J):\ \psi\in C_c^{\infty}(\mathcal{F}^{+})\ \textit{real}\big\},\qquad \mathcal{T}_7=\mathcal{T}_5\cup\mathcal{T}_{\mathcal{F}^{+}}.$$

Then $\mathcal{T}_7\supseteq\mathcal{T}_6$ is faithful, $v\mathbf{1}_{U_7}\in\mathcal{H}(\mathcal{T}_7)$ with $U_7=S_3\cup\mathcal{Z}\cup\mathcal{R}\cup J\mathcal{R}\cup\mathcal{F}^{+}\cup J\mathcal{F}^{+}$, and

$$\begin{aligned}\mu(U_7)&=C_0+2\mu\big((N_0\cap\mathcal{F}^{+})\setminus\mathcal{R}\big)\\&=\frac{1126668325607502584615015940142157374558020 61}{27056964762462501481857964726258382399631 36}\\&=41.64060290940640373015\ldots.\end{aligned}$$

Overview. The proof of Theorem 67 uses the shape of $\mathcal{P}$ only through Proposition 68, and Proposition 68 uses it only through Proposition 76. We prove the corresponding signed estimate on $\mathcal{P}^{+}$ (Proposition 81) and then repeat the remaining steps with $\mathcal{F}^{+}$ in place of $\mathcal{F}$ (Proposition 91 and Section 8.4). The proof of Proposition 81 follows that of Proposition 76 until the dual sum has been separated into four smooth variables, but keeps the sum over the moduli. If one variable is long, or the variables

can be split into two long groups, the estimates of Lemma 84 apply at each modulus. Otherwise two variables are long, and we transform both by Poisson summation (Lemma 85). The result is a sum of Kloosterman fractions $e_{u'}(\pm r\, y_0\, \overline{F_0\, e\, d_0\, b})$ over a reduced modulus u', which we bound in the four ways described above (Lemmas 86 to 89). The results quoted from the literature for these four bounds are collected, in the form used, in Section 8.1.

8.1 Quoted estimates

Besides the results already used in Sections 5 and 7, this section uses the following four statements, which we quote in the form and normalisation in which they are applied. Throughout, $\overline{x}$ denotes an inverse of x modulo the relevant modulus, $e_c(x) = e(x/c)$, and $S(m, n; c) = \sideset{}{^*}\sum_{x \bmod c} e_c(mx + n\overline{x})$.

(Q1) Sums of Kloosterman fractions [16, Theorem 12], with the third term as corrected in [28, Lemma 7.5]. Let $C, D, N, R, S \geq 1$, let g be a smooth function supported in $[C, 2C] \times [D, 2D]$ with $\partial_c^{\nu_1} \partial_d^{\nu_2} g(c, d) \ll C^{-\nu_1} D^{-\nu_2}$ for $0 \leq \nu_1, \nu_2 \leq 2$, and let $b_{n,r,s}$ be complex numbers supported on $N \leq n \leq 2N$, $R \leq r \leq 2R$, $S \leq s \leq 2S$. Then

$$\sum_c \sum_d \sum_{\substack{n,r,s \\ (rd, sc) = 1}} b_{n,r,s}\, g(c, d)\, e\Big(\frac{n\overline{rd}}{cs}\Big) \ll_\epsilon (CDNRS)^\epsilon\, \mathcal{K}\, \|b\|_2, \tag{49}$$

where $\overline{rd}$ is the inverse modulo cs, $\|b\|_2^2 = \sum |b_{n,r,s}|^2$, and

$$\mathcal{K}^2 = CS(RS + N)(C + DR) + C^2 DS\sqrt{(RS + N)R} + D^2 NR. \tag{50}$$

In [16, Theorem 12], and in its extension to moduli in arithmetic progressions [17, Theorem 2.1], the third term reads $D^2 NRS^{-1}$; [28, Lemma 7.5] corrects it to $D^2 NR$. The two agree for $S = 1$, which is the case used in Lemma 86. The right side of (49) is nondecreasing in C, D, N, R, S; this uses the weaker third term $D^2 NR$, since $D^2 NRS^{-1}$ is not nondecreasing in S. In particular it makes no difference whether the dyadic ranges are taken closed or half-open, and after a dyadic decomposition the same bound holds, with a factor $(\log CDNRS)^3$, when the coefficients are supported in $[1, 2N] \times [1, 2R] \times [1, 2S]$. Negative n (or d) are reduced to positive ones by complex conjugation (or by $d \mapsto -d$, $n \mapsto -n$).

(Q2) The spectral large sieve [16, Theorem 2], in the normalisation of (35). Let $A \geq 1$, let $\mathfrak{c}$ be a cusp of $\Gamma_0(A)$ equivalent to ∞ or to $1/m$ with $m \mid A$ and

$(m, A/m) = 1$ (so that $\mu(\mathfrak{c}) = 1/A$ in the notation of [16]), let $U, N \geq 1$, and let $(a_n)_{N<n\leq 2N}$ be complex. Then

$$\sum_{|t_j|\leq U} \Big| \sum_n a_n \rho_{j\mathfrak{c}}(n) \Big|^2 \ll_\epsilon (ANU)^\epsilon \Big(U^2 + \frac{N}{A} \Big) \sum_n |a_n|^2, \tag{51}$$

where $\rho_{j\mathfrak{c}} = \rho_{j\mathfrak{c}}^{\mathrm{raw}} / \sqrt{\cosh(\pi t_j)}$ for an orthonormal basis $\{u_j\}$ of Maass cusp forms of $\Gamma_0(A)$ (Petersson product not divided by the volume), the exceptional forms included; and the same bound holds for the holomorphic forms of weight at most U with the coefficients $\psi_{jk\mathfrak{c}}$ of (35), and for the Eisenstein series with $\frac{1}{4\pi} \sum_{\mathfrak{c}'} \int_{-U}^{U}$. In [16, Theorem 2] the raw coefficients carry the weight $1/\cosh(\pi t_j)$, which is our normalisation; at an exceptional parameter $t_j = i\tau_j$ this weight lies in $[1, 1.07]$ by the Kim–Sarnak bound $\tau_j \leq \frac{7}{64}$ [26]. The same inequality, with the same proof, was used in the proof of Proposition 45. For indices $-2N \leq n < -N$ apply (51) to the basis $u_j(-\bar{z})$, which is again orthonormal; the reflection maps the cusps ∞ and $1/m$ to equivalent cusps and changes the scaling matrices by a translation, which multiplies $\rho_{j\mathfrak{c}}(n)$ by a unimodular factor depending only on n.

(Q3) Exceptional spectrum with a weight, from [29, Lemma 3.9 and Corollary 3.10]. For $Y \geq 1$ and $(a_n)_{N<n\leq 2N}$, with the normalisation of (Q2) at the cusp ∞ of $\Gamma_0(A)$,

$$\sum_{t_j = i\tau_j \text{ exc}} Y^{2\tau_j} \Big| \sum_n a_n \rho_{j\infty}(n) \Big|^2 \ll_\epsilon (ANY)^\epsilon \Big(1 + \frac{N}{A} + \frac{N^{3/2} Y^{1/2}}{A} \Big) \sum_n |a_n|^2. \tag{52}$$

This is not stated as such in [29]; it follows from [29, Corollary 3.10] by the argument sketched in [29, §2]. With $X = Y$ and the cusp ∞ of $\Gamma_0(A)$, for which $\mu(\infty) = 1/A$, that corollary bounds the left side of (52) by a constant times

$$\sum_{c \equiv 0 \,(A)} \frac{1}{c} \sum_{m,n} |a_m a_n| \, |S(m, n; c)| \, \Phi\Big(\frac{Y\sqrt{mn}}{c} \Big) + O_\epsilon\Big(\Big(1 + \frac{N^{1+\epsilon}}{A} \Big) \|a\|_2^2 \Big),$$

where $\Phi \geq 0$ is a fixed smooth function supported on $t \asymp 1$. Here the moduli of the cusp ∞ are the $c \equiv 0 \pmod{A}$ [29, Lemma 3.2]; the regular spectrum, which gives the second term, is bounded by [29, Lemma 3.9], that is by [16, Theorem 2]; and the test function in the proof of the corollary is $\phi(y) = \Phi(yY/4\pi)$, so that only the moduli $c \asymp NY$ occur. (The printed statement of [29, Corollary 3.10] has $\sqrt{mn}/c$ as the argument of Φ; the factor Y comes from this choice of ϕ, and the bound below is the one obtained in [29, §2].)

The coefficients in [29] are those of [16], without the factor $\cosh(\pi t_j)^{-1/2}$, which at an exceptional parameter lies in $[1, 1.07]$ as noted in (Q2). For each modulus, Weil's bound (Q4) and $\sum_{m,n} |a_m a_n|(m,c)^{1/2} \ll c^\epsilon N \|a\|_2^2$ bound the double sum of Kloosterman sums by $c^{1/2+\epsilon} N \|a\|_2^2$, and summing over the $O(NY/A)$ moduli $c \equiv 0 \pmod A$ with $c \asymp NY$, each with the factor c^{-1}, gives $(ANY)^\epsilon N^{3/2} Y^{1/2} A^{-1} \|a\|_2^2$. We use (52) only with $Y = A^2/N^3 \geq 1$, where its right side is $\ll (ANY)^\epsilon \|a\|_2^2$. As in (Q2), it holds also at a cusp $1/m$ with $m \mid A$, $(m, A/m) = 1$, since by Lemma 42 the coefficients of u_j at this cusp are those of $u_j | W_{A/m}$ at ∞, and $\{u_j | W_{A/m}\}$ is an orthonormal basis of the same eigenspaces.

(Q4) Weil's bound [25, Corollary 11.12]: $|S(m,n;c)| \leq \tau(c)(m,n,c)^{1/2} c^{1/2}$ for all integers m, n and $c \geq 1$; for $m = 0$ this includes the Ramanujan sum.

We also use the Kuznetsov formula for the cusp pair $(\infty, \mathfrak{a})$ exactly as in (35), with arbitrary nonzero integers m', n' in place of $1, \pm n$:

$$\begin{aligned} \sum_\gamma \frac{S_{\infty\mathfrak{a}}(m',n';\gamma)}{\gamma} \phi\Big(\frac{4\pi\sqrt{|m'n'|}}{\gamma}\Big) = {} & \sum_j \overline{\rho_{j\infty}(m')} \rho_{j\mathfrak{a}}(n') \widetilde{\phi}(t_j) \\ & + \mathbf{1}_{m'n'>0} \sum_{k,j} \widetilde{\phi}(k) \overline{\psi_{jk\infty}(m')} \psi_{jk\mathfrak{a}}(n') \\ & + \frac{1}{4\pi} \sum_{\mathfrak{c}} \int \overline{\rho_{\mathfrak{c}\infty}(m',t)} \rho_{\mathfrak{c}\mathfrak{a}}(n',t) \widetilde{\phi}(t)\, \mathrm{d}t, \end{aligned} \tag{53}$$

where γ runs over the moduli of the pair, the transforms are those of [16, Theorem 1] for the sign of $m'n'$, and for ϕ supported on $\xi \asymp \xi_0 \leq 1$ with $\xi^i \phi^{(i)}(\xi) \ll_i \Phi_0$ they satisfy (36) with $X^\epsilon Q\sqrt{A}$ replaced by $(\Phi_0/\xi_0)^\epsilon \Phi_0$ and K_B by ξ_0^{-1}. In this direction (Kloosterman sums to spectrum) the formula has no diagonal term, whether or not the two cusps are equivalent. The proof of Lemma 42 gives, for all integers m', n',

$$S_{\infty\mathfrak{a}}(m', n'; c\sqrt{a}) = S(m', n'\overline{a}; c) \qquad (m \mid c,\ (c,a) = 1), \tag{54}$$

since there the Kloosterman phase is $e((m'x + n'y)/c)$ with $y \equiv \overline{a}\,\overline{x} \pmod c$ and x running once over $(\mathbb{Z}/c\mathbb{Z})^*$.

8.2 The four-shift row on $\mathcal{P}^+$

Proposition 80 (Four-shift row)**.** *The conclusion of Proposition 68 holds for every $F \in C_c^\infty(\mathbb{R}^2)$ with support in a compact subset of $\mathcal{P}^+$.*

Proof. We follow the proof of Proposition 68, with $\kappa_+ = \max\{a_1 + a_2 : a \in \operatorname{supp} F\} < \frac{3}{5}$ in place of $\max\{a_1 + a_2\} < \frac{13}{25}$. The stationary-phase formula (15) uses only $\kappa_+ < 1$, so it holds with error $O(T^{\kappa_+ + \epsilon})$, which is $O(T^{\frac{3}{5}})$ for $\epsilon < \frac{3}{5} - \kappa_+$, and its main term is $2\pi \sum_{h,k} \overline{F(\log h/L, \log k/L)}(hk)^{-1}\mathcal{S}(hk)$. In the character expansion (44) we treat the three parts as there. For the principal characters off the poles, the new line contributes $\ll T^{\frac{1}{2}+\delta}N^{2\epsilon}$ for each $N = hk$, as shown in the proof of Proposition 68; with the weight $1/N$ and $N \le T^{\kappa_+}$ the total is $O(T^{\frac{1}{2}+2\delta})$, whatever $\kappa_+ < 1$. The residue computation does not use the size of κ_+. The non-principal characters contribute

$$2\pi \sum_{h,k\ge 1} \frac{\overline{F(\log h/L, \log k/L)}}{hk} \mathcal{S}^\flat(hk) \ll T^{1-\eta}$$

by Proposition 81. □

Proposition 81. *Let $F \in C_c^\infty(\mathbb{R}^2)$ have support in a compact set $E \subset \mathcal{P}^+$. There is $\eta > 0$ such that, uniformly for $\boldsymbol{\sigma}$ in compact subsets of $(i\mathbb{R})^4$ and Φ in bounded subsets of $\mathcal{W}$,*

$$\Sigma_F := \sum_{h,k\ge 1} \frac{\overline{F(\log h/L, \log k/L)}}{hk} \mathcal{S}^\flat(hk) \ll T^{1-\eta}.$$

Unlike Proposition 76, this bounds a signed sum: in the cases of Lemmas 87 and 89 the cancellation comes from the sum over the moduli, with the weight F evaluated at the factorisation $q = hk$. The proof occupies the rest of this subsection and is completed after Lemma 90.

The signed sum and its conductor labels

Insert Lemma 69 for each $q = hk$, and note that $d^2\varphi(d)/\varphi(dr) = d^2/\varphi(r)$ because $(d, r) = 1$. We first record how the terms depend on the primitive conductor d.

Lemma 82 (Conductor labels). *In Lemma 69, $b_\ell(g, d, r) = b_\ell^\circ(g, r)\mathbf{1}_{(\ell,d)=1}$, where $b_\ell^\circ(g, r)$ depends only on g, r and $\boldsymbol{\alpha}$, vanishes unless $\ell \mid g^3r^4$, and satisfies $\sum_\ell |b_\ell^\circ(g, r)| \ll_\epsilon (gr)^\epsilon$. With $q = gdr$ and $X = qT/(2\pi)$,*

$$Y = \frac{g\ell d^4}{\pi^4 X} = \frac{2\ell d^3}{\pi^3 rT}, \qquad \left|\frac{d^2\varpi_d}{\varphi(r)}\right| = \frac{d^2}{\pi^2\varphi(r)}, \qquad \varpi_d = \pi^{-2+\sum\alpha_i} d^{-\sum\alpha_i}.$$

Proof. In the proof of Lemma 69, for each prime $p \mid gr$ let $v = v_p(g)$ and put

$$P_p(X) = \begin{cases} d_{\boldsymbol{\alpha}}(p^v)\prod_{i=1}^{4}(1-p^{-\alpha_i}X), & p \mid r, \\ \prod_{i=1}^{4}(1-p^{-\alpha_i}X)\sum_{j\ge 0} d_{\boldsymbol{\alpha}}(p^{v+j})X^j, & p \nmid r, \end{cases}$$

a polynomial of degree at most 4 (if $p \mid r$) or 3 (if $p \nmid r$), whose coefficients depend only on p, v and $\boldsymbol{\alpha}$. The local factor of $D(w)$ divided by that of $\prod_i L(w+\alpha_i,\psi)$ is $P_p(\psi(p)p^{-w})$ for every $p \mid gr$: for $p \mid r$ and for $p \mid g$, $p \nmid q'$ this is the computation in that proof, and for $p \mid g$, $p \mid d$ we have $\psi(p) = 0$, the local factor of the L-functions is 1, and $P_p(0) = d_{\boldsymbol{\alpha}}(p^v)$ is the constant local factor of $D(w)$ found there. At $p \nmid gr$ the two local factors agree. Hence $\sum_\ell b_\ell \psi(\ell)\ell^{-w} = \prod_{p\mid gr} P_p(\psi(p)p^{-w}) = \sum_\ell b^\circ_\ell(g,r)\psi(\ell)\ell^{-w}$, where $b^\circ_\ell(g,r)$ are the coefficients of the expanded product, which is independent of d; and $\psi(\ell) = 0$ unless $(\ell, d) = 1$. The support and the bound follow as in Lemma 69. The formula for Y is $X = gdrT/(2\pi)$ substituted, and $|\varpi_d| = \pi^{-2}$ because the α_i are imaginary. □

Consequently, writing $\epsilon_{\mathfrak{a}}(\pm) = \frac{1}{2}(\pm 1)^{\mathfrak{a}}$ for the parity projector,

$$\begin{aligned} \Sigma_F = \sum_{g,r}\frac{\mu(r)}{\varphi(r)}\sum_{\ell} b^\circ_\ell(g,r)\sum_{\mathfrak{a},\pm}\epsilon_{\mathfrak{a}}(\pm)\sum_{\substack{d>1\\(d,r\ell)=1}} d^2\varpi_d \\ \times \sum_{hk=gdr}\frac{\overline{F(\log h/L,\log k/L)}}{hk}\sum_{n\ge 1}\frac{d_{-\boldsymbol{\alpha}}(n)}{n}\,V_{\mathfrak{a}}\Big(\frac{n}{Y}\Big)\mathscr{K}_d\big(\pm rn\bar{\ell}\big), \end{aligned} \tag{55}$$

with r squarefree. At fixed labels $(g, r, \ell, \mathfrak{a}, \pm)$ the summand depends on d only through the power $d^{2-\sum\alpha_i}$, the cut $(d, r\ell) = 1$, the smooth weight $V_{\mathfrak{a}}(n/Y)$ with $Y \propto d^3$, the kernel $\mathscr{K}_d$ with the unit dilation $\pm r\bar{\ell}$, and the modulus weight at the factorisations $hk = gdr$. No coefficient depends on d in any other way; this is what allows us to sum over d with the signs kept.

Separation of variables. Insert smooth dyadic partitions of unity in h, k, d and in each of the four variables of $d_{-\boldsymbol{\alpha}}(n) = \sum_{n_1n_2n_3n_4=n}\prod_i n_i^{\alpha_i}$, and dyadic ranges for the labels g, r, ℓ. On a box $h \asymp H$, $k \asymp K$ the function $\overline{F(\log h/L, \log k/L)}W(h/H)W(k/K)$ is a smooth function of $(\log h, \log k)$ supported in a fixed square, with all derivatives bounded (each derivative of $F(\cdot/L)$ gains $1/L$); by Fourier inversion in $(\log h, \log k)$ it is $\iint G(\theta_1,\theta_2)h^{i\theta_1}k^{i\theta_2}\,\mathrm{d}\theta$ with $\iint |G| \ll 1$. By the definition of $V_{\mathfrak{a}}$,

$$V_{\mathfrak{a}}\Big(\frac{n}{Y}\Big) = \frac{1}{2\pi i}\int_{(c)}\widetilde{\Phi}(w)\,G_{\mathfrak{a}}(w)\Big(\frac{\pi^3 rT}{2\ell}\Big)^w d^{-3w}n^w\,\mathrm{d}w,$$

with $G_{\mathfrak{a}}$ the gamma quotient, on any line $c < \frac{19}{20}$ (proof of Lemma 70). On a box $n_i \asymp N_i$, $d \asymp D_1$ we take $c = \frac{9}{10}$ if $N_* := \prod N_i \le 16Y_1$, where Y_1 is the value of Y at $d = D_1$, and $c = -A$ otherwise; the part with $|\Im w| > Q^{\epsilon}$ is negligible. After these insertions every weight is a product of functions of the single variables $h, k, d, n_1, \dots, n_4$, each of the form $W(y/Y_0)\, y^{\xi}$ with W a fixed smooth bump and $|\Im \xi| \le Q^{\epsilon}$, so with rescaled derivatives $\ll Q^{j\epsilon}$; the integrals over θ and w cost $Q^{O(\epsilon)}$. Here and below $Q = T^{\kappa}$ with κ the value of $\log q/L$ at a point of E near which F is supported; by a partition of unity we may assume that $\operatorname{supp} F$ lies in a small ball around that point, so that $\log q/\log Q = 1 + O(\epsilon)$ for all $q = hk$ in the support. We write $Q^{o(1)}$ for factors $Q^{O(\epsilon)}$ and $(\log Q)^{O(1)}$.

Degrees. All degrees are to base Q: $[x] = \log x/\log Q$, taken at the dyadic scale. For a dyadic datum $\mathfrak{D}$ (the boxes of $h, k, g, r, \ell, d, n_1, \dots, n_4$) put $\mathfrak{g} = [g]$, $\mathfrak{r} = [r]$, $\mathfrak{i} = [\ell]$, let $\varsigma = 2 - L/\log q$ and $\mathfrak{b} = \log h/\log q$ at the box, and let $\mathfrak{v} = \max(0, [Y_1/N_*])$ be the tail degree. Put

$$\sigma = 1 - \mathfrak{g} - \mathfrak{r} = [d], \qquad \lambda = 1 + \varsigma - 3\mathfrak{g} - 4\mathfrak{r} - \mathfrak{v},$$
$$\tau = \varsigma - 2\mathfrak{g} - 3\mathfrak{r} - \tfrac{9}{10}\mathfrak{v}, \qquad \mathsf{B}_* = \tfrac{1}{2}\sigma + 2\tau.$$

Order the four dual degrees $[N_i]$, let $\mathsf{m} \ge \mathsf{n}$ be the two largest and j the sum of the other two, and put $\mathsf{D} = \mathsf{m} + \mathsf{n} + \mathsf{j}$. For a modulus q let $P(q)$ be its powerful part and $\mathfrak{p} = [P(q)]$, and put $\tau_{\mathfrak{p}} = \tau - \frac{1}{2}\mathfrak{p}$; in Proposition 76 the letter τ denoted $\tau_{\mathfrak{p}}$.

Lemma 83 (Degrees and normalisation)**.** (i) *Up to an error that tends to* 0 *with* ϵ*,*

$$\mathsf{D} = \lambda + \mathfrak{i} = 1 + \varsigma - 3\mathfrak{g} - 4\mathfrak{r} - \mathfrak{v} + \mathfrak{i}, \qquad 0 \le \mathfrak{i} \le 3\mathfrak{g} + 4\mathfrak{r},$$

so $\lambda \le \mathsf{D} \le 1 + \varsigma$*; boxes with* $N_* > Y_1 Q^{\epsilon}$ *contribute* $O(T^{-A})$*.*

(ii) *Let* $\mathscr{B}_{\mathfrak{D}}(q)$ *be the contribution of the datum* $\mathfrak{D}$ *to* $\mathcal{S}^{\flat}(q)$*, and* $\mathscr{B}_{\mathfrak{D}} = \sum_{h,k} \overline{F(\cdot)}(hk)^{-1}\mathscr{B}_{\mathfrak{D}}(hk)$ *its contribution to* Σ_F*. Normalise the sum over* n *by* N_* *and the sum over the moduli by their number. Then the trivial bound is* $|\mathscr{B}_{\mathfrak{D}}| \le Q^{2-2\mathfrak{g}-3\mathfrak{r}-\frac{9}{10}\mathfrak{v}+o(1)}$*, and* $\mathscr{B}_{\mathfrak{D}} \ll T^{1-\delta}$ *holds as soon as*

- *(a) the normalised sum over* d *and* n *(all moduli of the datum together) is* $\ll Q^{-\tau-\delta'}$*, or*
- *(b) for every modulus* q *of the datum, the normalised sum over* n *at that modulus is* $\ll Q^{-\tau_{\mathfrak{p}(q)}-\delta'}$*,*

for some $\delta' > 0$*, where in (b)* δ *depends on* δ' *only.*

Proof. (i) By Lemma 82, $[Y_1] = \mathfrak{i} + 3\sigma - \mathfrak{r} - (2 - \varsigma) = 1 + \varsigma - 3\mathfrak{g} - 4\mathfrak{r} + \mathfrak{i}$, since $T = q^{2-\varsigma}$. If $N_* \le 16Y_1$ then $[N_*] = [Y_1] - \mathfrak{v} + o(1)$. If $N_* > 16Y_1$ then $\mathfrak{v} = 0$,

the line $c=-A$ gives the factor $(N_*/Y_1)^{-A}$, so $N_*\le Y_1Q^{\epsilon}$ up to $O(T^{-A})$, and $[N_*]=[Y_1]+O(\epsilon)$. The bound $\mathsf{i}\le 3\mathfrak{g}+4\mathfrak{r}$ is $\ell\mid g^3r^4$, and $\mathsf{D}\le 1+\varsigma$ follows.

(ii) On the box, $|n^{-1}V_{\mathfrak{a}}(n/Y)|\ll N_*^{-1}(N_*/Y_1)^{9/10}\le N_*^{-1}Q^{-\frac{9}{10}\mathfrak{v}}$ for $c=\frac{9}{10}$, and $\ll N_*^{-1}$ for $c=-A$ (where $\mathfrak{v}=0$); this is Lemma 70. With $|\mathscr{K}_d|\le q^{o(1)}$ (Lemma 71) the n-sum is at most $Q^{-\frac{9}{10}\mathfrak{v}+o(1)}$ times its normalisation, and $d^2/\varphi(r)\le Q^{2\sigma-\mathfrak{r}+o(1)}=Q^{2-2\mathfrak{g}-3\mathfrak{r}+o(1)}$. Summing over the $q^{o(1)}$ labels and factorisations of each q with the weight $1/q$, over $q\le T$, gives the trivial bound, and $Q^{2-\varsigma}=T$ gives (a). For (b), the hypothesis gives $|\mathscr{B}_{\mathfrak{D}}(q)|\ll Q^{2-\varsigma+\frac{1}{2}\mathfrak{p}-\delta'+o(1)}=TP(q)^{1/2}q^{-\delta'+o(1)}$, and, as at the end of the proof of Proposition 76, $\sum_{q\le T}P(q)^{1/2}q^{-1}\ll(\log T)^2$, the at most $\tau(q)=T^{o(1)}$ pairs (h,k) for each q, and $q\ge T^{1/2}$ give $\mathscr{B}_{\mathfrak{D}}\ll T^{1-\delta'/2+o(1)}$. □

The number of data $\mathfrak{D}$ is $(\log T)^{O(1)}$, so it suffices to prove (a) or (b) for each $\mathfrak{D}$ with a δ' that is uniform in $\mathfrak{D}$. Whether (a) or (b) is used, and which estimate below proves it, is decided by the limiting degrees of $\mathfrak{D}$ alone; in particular the decision does not depend on $\mathfrak{p}$, nor on the modulus within $\mathfrak{D}$. If $\tau<0$ (strictly, at the limiting tuple) the trivial bound gives (a).

Estimates at each modulus

Lemma 84 (Unsigned estimates). *Let $d>1$, let c be a unit modulo d, and let P be the powerful part of d. Let W be smooth as in Section 7.*

(i) *For $M\ge 1$, $\frac{1}{M}\big|\sum_m W(m/M)\mathscr{K}_d(cm)\big|\ll q^{o(1)}\big(\sqrt{dP}/M+\sqrt{P/d}\big)$.*

(ii) *For $M,N\ge 1$ and coefficients $|\alpha_m|,|\beta_n|\le q^{o(1)}$ supported on $m\in[M,2^8M]$, $n\in[N,2^8N]$,*

$$\Big(\frac{1}{MN}\Big|\sum_{m,n}\alpha_m\beta_n\mathscr{K}_d(cmn)\Big|\Big)^2\ll q^{o(1)}\Big(\frac{1}{N}+\frac{\sqrt{dP}}{M}+\sqrt{\frac{P}{d}}\Big).$$

Consequently the normalised dual sum of a datum with $\tau\ge 0$ is $\ll Q^{-\tau_{\mathfrak{p}}-\delta'}$ at every modulus, which is (b) of Lemma 83, if either $\mathsf{m}>\frac{1}{2}\sigma+\tau$, or $\sigma>4\tau$ and the four dual variables split into two groups of degrees $\mathsf{s}\le\mathsf{l}$ with $\mathsf{s}>2\tau$ and $\mathsf{l}>\mathsf{B}_*$ (each strictly, at the limiting tuple).

Proof. (i) Let H' be the product of the prime powers $p^a\|d$ with $a\ge 2$, and $K'=d/H'$, which is squarefree and coprime to H'; so $H'\le P$. By Poisson summation modulo d,

$$\sum_m W(m/M)\mathscr{K}_d(cm)=\frac{M}{d}\sum_{j\in\mathbb{Z}}\widehat{W}(jM/d)\widehat{\mathscr{K}}(j),$$

where $\widehat{\mathscr{K}}(j) = \sum_{x \bmod d} \mathscr{K}_d(cx)e_d(jx)$. By Lemma 71(i) and the Chinese remainder theorem, $\widehat{\mathscr{K}}(j)$ is a product of complete sums modulo the prime powers of d. The factor at $t \mid H'$ is at most $9t$ by Lemma 71(iii). The factor at $p \mid K'$ is at most $2\sqrt{p}+1$, and vanishes if $p \mid j$, by Lemma 71(iv), which holds at every prime. Hence $\widehat{\mathscr{K}}(0) = 0$ if $K' > 1$, and $|\widehat{\mathscr{K}}(j)| \le q^{o(1)}H'\sqrt{K'}$. Since $\frac{M}{d}\sum_{j\neq 0}|\widehat{W}(jM/d)| \ll q^{o(1)}(1+M/d)$, the left side of (i) is $\ll q^{o(1)}(H'\sqrt{K'}/M + H'\sqrt{K'}/d)$, and $H'\sqrt{K'} = \sqrt{dH'} \le \sqrt{dP}$, $H'\sqrt{K'}/d = \sqrt{H'/d} \le \sqrt{P/d}$. If $K' = 1$ then $d \le P$ and (i) is trivial.

(ii) By Cauchy's inequality in m, with a smooth $W \ge 0$ equal to 1 on $[1, 2^8]$, the square of the sum is at most $q^{o(1)}M\sum_{n_1,n_2}|\beta_{n_1}\beta_{n_2}||\mathcal{C}(n_1,n_2)|$ with $\mathcal{C}(n_1,n_2) = \sum_m W(m/M)\mathscr{K}_d(cmn_1)\overline{\mathscr{K}_d(cmn_2)}$. The terms $n_1 = n_2$ contribute $q^{o(1)}M^2N$, which is the term $1/N$. For $n_1 \neq n_2$ complete the m-sum modulo d as in (i): $\mathcal{C} = \frac{M}{d}\sum_j \widehat{W}(jM/d)\widehat{\mathcal{C}}(j)$, where $\widehat{\mathcal{C}}(j)$ factors over the prime powers $t\|d$ into the complete sums $\sum_{x \bmod t}\mathscr{K}_t(c_t x n_1)\overline{\mathscr{K}_t(c_t x n_2)}e_t(j_t x)$ with units c_t. If $p \mid n_1n_2$ for some $p \mid d$ the summand vanishes identically. Let K'' be the product of the primes $p \ge 5$ with $p\|d$, and $H'' = d/K'' \le 6P$. At $p \mid K''$ with $p \nmid n_1n_2(n_1^2 - n_2^2)$, Lemma 72 with $m = 2$, $A_1(x) = c_pn_1x$, $A_2(x) = c_pn_2x$ (equal roots, slopes $c_pn_1 \neq \pm c_pn_2$) bounds the factor by $C\sqrt{p}$; at the other primes of K'' we use the bound $16p$, and at $t \mid H''$ the bound $81t$. So $|\widehat{\mathcal{C}}(j)| \le q^{o(1)}H''\sqrt{K''}(K'', n_1^2 - n_2^2)^{1/2}$, and $H''\sqrt{K''} = \sqrt{dH''} \le \sqrt{6dP}$. Hence

$$\frac{|\mathcal{C}(n_1,n_2)|}{M} \ll q^{o(1)}\Big(\frac{\sqrt{dP}}{M} + \sqrt{\frac{P}{d}}\Big)(K'', n_1-n_2)^{1/2}(K'', n_1+n_2)^{1/2},$$

using that K'' is squarefree. By $xy \le \frac{1}{2}(x^2+y^2)$ and $\sum_{0<|a|\le A}(K'', a) \le \tau(K'')\cdot 2A$, the sum of the last two factors over $n_1 \neq n_2$ is $\ll q^{o(1)}N^2$. Collecting the terms gives (ii). No condition on the sizes of the factors of q is used.

For the consequences, let $Q^{\mathfrak{p}}$ be the powerful part of the modulus; the powerful part of d is at most that of q. In (i), with $M = Q^{\mathsf{m}}$ and the other three variables fixed (the product with the unit c is a unit or the inner sum vanishes), the saving is $\min(\mathsf{m} - \frac{1}{2}(\sigma+\mathfrak{p}), \frac{1}{2}(\sigma-\mathfrak{p}))$, which exceeds $\tau_{\mathfrak{p}} = \tau - \frac{1}{2}\mathfrak{p}$ if $\mathsf{m} > \frac{1}{2}\sigma + \tau$ and $\sigma > 2\tau$. Here $\sigma - 2\tau = 1 - 2\varsigma + 3\mathfrak{g} + 5\mathfrak{r} + \frac{9}{5}\mathfrak{v} > 0$ for $\varsigma < \frac{1}{2}$. In (ii), grouping the variables, with divisor-bounded coefficients, into m of degree l and n of degree s, the squared saving is $\min(\mathsf{s}, \mathsf{l} - \frac{1}{2}(\sigma+\mathfrak{p}), \frac{1}{2}(\sigma-\mathfrak{p}))$, which exceeds $2\tau_{\mathfrak{p}} = 2\tau - \mathfrak{p}$ if $\mathsf{s} > 2\tau$, $\mathsf{l} > \mathsf{B}_*$ and $\sigma > 4\tau$, for every $\mathfrak{p} \ge 0$. The identity $\sigma - 4\tau = 1 - 4\varsigma + 7\mathfrak{g} + 11\mathfrak{r} + \frac{18}{5}\mathfrak{v}$ shows $\sigma > 4\tau$ for $\varsigma < \frac{1}{4}$; for $\frac{1}{4} \le \varsigma < \frac{1}{3}$ the failure $4\tau \ge \sigma$ is one of the alternatives in Lemma 90(iii). □

Remark. In Lemmas 73 and 74 the primes of h were put into H in order to shift by multiples of H; the form above completes directly modulo d, puts only the powerful part (and in (ii) the primes $2, 3$) into H', H'', and needs no condition on the sizes of the factors of q. The thresholds in the last paragraph do not depend on $\mathfrak{p}$; the density of powerful moduli enters only through Lemma 83(ii)(b).

Double Poisson

In the remaining cases $\mathsf{m} \le \frac{1}{2}\sigma + \tau$ and no split as in Lemma 84 succeeds. Let n_1, n_2 be the variables of degrees $\mathsf{m} \ge \mathsf{n}$, of lengths $U_m = Q^{\mathsf{m}}$, $U_n = Q^{\mathsf{n}}$, and keep the other two in their product $t = n_3 n_4 \asymp Q^{\mathsf{j}}$, whose coefficient γ_t (a convolution of two separated weights) is divisor-bounded and independent of d.

Lemma 85 (Double Poisson)**.** *Let $d > 1$, let $d_{(1)}$ be the product of the primes p with $p \| d$, and let c, t be units modulo d. For integers a, b,*

$$\frac{1}{d} \sum_{x_1, x_2 \bmod d} \mathscr{K}_d(c\,t\,x_1 x_2)\, e_d(b x_1 + a x_2) = \mathbf{1}_{(ab,d)=1} \sum_{l \mid d_{(1)}} \frac{1}{\varphi(l)}\, e_{d/l}\big(c\,t\,\bar{l}\,\overline{ab}\big).$$

Consequently

$$\sum_{n_1, n_2} W_1\Big(\frac{n_1}{U_m}\Big) W_2\Big(\frac{n_2}{U_n}\Big) \mathscr{K}_d(c t n_1 n_2) = \frac{U_m U_n}{d} \sum_{(ab,d)=1} \widehat{W}_2\Big(\frac{aU_n}{d}\Big) \widehat{W}_1\Big(\frac{bU_m}{d}\Big) \sum_{l \mid d_{(1)}} \frac{e_{d/l}(ct\bar{l}\,\overline{ab})}{\varphi(l)},$$

the dual variables have effective lengths of degrees $\mathsf{a} = \sigma - \mathsf{n}$ and $\mathsf{b} = \sigma - \mathsf{m}$, both positive, and the trivial bound of the term l, normalised as in Lemma 83, exceeds the required $Q^{-\tau}$ by $Q^{\mathsf{j}+\mathfrak{v}/10-\mathsf{i}-\mathsf{z}+o(1)}$, where $\mathsf{z} = [l]$.

Proof. By Lemma 71(i) and the Chinese remainder theorem, both sides factor over the prime powers $t' \| d$ (with the dilations $\overline{(d/t')}$, which combine as $\prod_{t'} e_{t'}(y\overline{(d/t')}) = e_d(y)$), so it suffices to take $d = p^k$. For a primitive ψ modulo p^k and an integer a, $\sum_x \bar{\psi}(x) e_{p^k}(ax)$ is $\psi(a)\tau(\bar{\psi})$ if $p \nmid a$ and 0 if $p \mid a$. Hence the left side is 0 unless $p \nmid ab$, and then equals

$$\frac{1}{p^k \cdot p^k \varphi(p^k)} \sideset{}{^*}\sum_{\psi} \tau(\psi)^3 \tau(\bar{\psi})^2 \bar{\psi}(ct)\psi(ab) = \frac{1}{\varphi(p^k)} \sideset{}{^*}\sum_{\psi} \tau(\psi)\bar{\psi}(y), \qquad y = ct\overline{ab},$$

using $\tau(\psi)\tau(\bar\psi)=\psi(-1)p^k$. Since $\sum_{\psi \bmod p^k}\tau(\psi)\bar\psi(y)=\varphi(p^k)e_{p^k}(y)$, and the imprimitive characters have $\tau(\psi)=0$ if $k\ge 2$ (proof of Lemma 71) while for $k=1$ the principal character has $\tau=-1$, the right side is $e_{p^k}(y)$ if $k\ge 2$ and $e_p(y)+\frac{1}{p-1}$ if $k=1$. Multiplying over p and expanding the product gives the sum over $l\mid d_{(1)}$, with $e_{d/l}(y\bar l)$ as the product of the remaining additive characters. The second display follows by Poisson summation modulo d in n_1 and n_2 (with $\widehat W(\xi)=\int W(y)e(-y\xi)\,\mathrm{d}y$). The transforms $\widehat{W_i}$ are Schwartz, so $|a|\le Q^{\epsilon}d/U_n$, $|b|\le Q^{\epsilon}d/U_m$ up to $O(T^{-A})$. Here $\mathsf{a}\ge\mathsf{b}=\sigma-\mathsf{m}\ge\frac12\sigma-\tau>0$. After division by U_mU_n, the term l is trivially at most $Q^{o(1)}d^{-1}\varphi(l)^{-1}(d/U_n)(d/U_m)=Q^{\sigma-\mathsf{m}-\mathsf{n}-\mathsf{z}+o(1)}$, and $\tau+\sigma-\mathsf{m}-\mathsf{n}=\mathsf{j}+\frac{1}{10}\mathfrak{v}-\mathfrak{i}$ by Lemma 83(i). □

Reduction of the modulus. Write $d=lu$ with $l\mid d_{(1)}$, so that l is squarefree and $(l,u)=1$, and fix l. The transform requires $(t,d)=1$ (otherwise $\mathscr{K}_d(ct\,\cdot)\equiv 0$), which is $(t,l)=1$ and $(t,u)=1$. Expand the second condition by Möbius inversion *before* any absolute value is taken: $\mathbf{1}_{(t,u)=1}=\sum_{f\mid(t,u)}\mu(f)$, and write $u=fu'$, $t=fy_0$. Since $t=fy_0$ and $u=fu'$, every term satisfies $e_u(ct\bar l\,\overline{ab})=e_{u'}(cy_0\bar l\,\overline{ab})$, the inverses now taken modulo u' (an inverse modulo u is also one modulo u'). The conditions $(ab,d)=1$ become $(ab,u')=1$ and $(ab,lf)=1$. Expand $(a,lf)=1$ by Möbius inversion as well: $a=ed_0$ with $e\mid lf$. Keep $(b,lf)=1$ as a cut in the weight of b, and $(fy_0,l)=1$ in that of y_0. With $c=\pm r\bar\ell$ and $F_0=l\ell$ the phase becomes

$$e_{u'}\big(\pm\, r\,y_0\,\overline{F_0\,e\,d_0\,b}\big)\qquad\text{on}\quad (rF_0ed_0b,u')=1, \tag{56}$$

where $(r\ell,u')=1$ comes from $(d,r\ell)=1$, $(l,u')=1$ from $(l,u)=1$, and $(ed_0b,u')=1$ from $(ab,u')=1$. Here $y_0\ge 1$, since $t\ge 1$. Put $\mathsf{x}=[f]$, $\mathsf{E}=[e]$, $\mathsf{Z}=\mathsf{z}+\mathsf{x}$, $\mathsf{y}=\mathsf{j}-\mathsf{x}$ and $\mathsf{C}=\sigma-\mathsf{Z}=[u']$; then $\mathsf{E}\le\min(\mathsf{Z},\mathsf{a})$, d_0 has effective length of degree $\mathsf{a}-\mathsf{E}$, and b of degree b. For fixed (l,f,e) let $\Sigma(l,f,e)$ be the corresponding part of the normalised sum of Lemma 83(ii)(a), all moduli $d=lfu'$ of the datum together. Its terms are

$$\Sigma(l,f,e)=\frac{\mu(f)\mu(e)}{\varphi(l)}\sum_{u'}\sum_{y_0}\sum_{d_0}\sum_{b}\varpi(u',y_0,d_0,b)\,e_{u'}\big(\pm\, r\,y_0\,\overline{F_0ed_0b}\big), \tag{57}$$

summed under (56), where the weight ϖ, of size $Q^{o(1)}$ times a normalising constant, is the product of: the modulus weight $\Theta(u')=\sum_{hk=grlfu'}\omega_h(h)\omega_k(k)$ with the separated mark weights ω_h,ω_k; the separated weights of $d=lfu'$ and the cut $(d,r\ell)=1$; the coefficient $\gamma_{fy_0}\mathbf{1}_{(fy_0,l)=1}$;

$\widehat{W}_2(ed_0U_n/d)$; and $\widehat{W}_1(bU_m/d)\mathbf{1}_{(b,lf)=1}$. On dyads of $u', |d_0|, |b|$ the last two factors separate into products of functions of single variables by Fourier inversion in the logarithms, with the rescaled derivative bounds of the Schwartz seminorms (also as the arguments tend to 0), at a cost $Q^{o(1)}$; we do this in Lemmas 86 to 88 but not in Lemma 89.

Normalisation. The factor $1/\varphi(l)$ saves z; for fixed f the variable y_0 has $Q^{\mathsf{j}-\mathsf{x}}$ values instead of Q^{j}, so the f-dyad saves x after its Q^{x} terms are summed; and the substitution $a = ed_0$ saves E. Summed over the dyads of l and f (which have harmonic weights in the average over the moduli, as $d = lfu'$), the trivial bounds of the $\Sigma(l,f,e)$ add up to $Q^{\sigma-\mathsf{m}-\mathsf{n}-\mathsf{Z}-\mathsf{E}+o(1)}$. Hence (a) of Lemma 83 holds for the datum if, for every choice of the dyads of l, f and of e, the sum of $\Sigma(l,f,e)$ over these dyads is smaller than its trivial bound by $Q^{\mathsf{u}+\delta'}$, where

$$\mathsf{u} = \tau + \sigma - \mathsf{m} - \mathsf{n} - \mathsf{Z} - \mathsf{E} = \mathsf{j} + \tfrac{1}{10}\mathfrak{v} - \mathsf{i} - \mathsf{Z} - \mathsf{E}.$$

If $\mathsf{u} < 0$ the trivial bound suffices. The trivial bound of the raw sum (57) at fixed (l,f,e), with ϖ replaced by its size, is $Q^{\mathsf{C}+\mathsf{y}+(\mathsf{a}-\mathsf{E})+\mathsf{b}+o(1)}$, so the requirement is that the raw sum be $\ll Q^{\mathsf{B}-\delta'}$ with

$$\mathsf{B} = \mathsf{C} + \mathsf{y} + \mathsf{a} - \mathsf{E} + \mathsf{b} - \mathsf{u} = 2\sigma + \mathsf{y} - \tau.$$

Four estimates for the Kloosterman fractions

Below, *margin* means the allowed exponent minus the exponent of the bound; an estimate saves a power when its margin is positive at the limiting tuple. All exponents are computed with the largest dual scales $Q^{\mathsf{a}-\mathsf{E}}$, Q^{b}; every bound used is nondecreasing in these scales, so smaller dyads of d_0 and b are covered. Put $\mathsf{c} = \mathsf{C} - \mathsf{w}$, $\mathsf{d} = \mathsf{a} - \mathsf{E}$, $\mathsf{R} = \mathsf{b} + \mathfrak{r} + \mathsf{z} + \mathsf{i} + \mathsf{E}$, $\mathsf{N} = 2\mathfrak{r} + \mathsf{y}$, where $\mathsf{w} = [S_0]$ is defined in Lemma 87. The identities between affine forms used below have also been checked by computer expansion.

Lemma 86 (Spectral square)**.** *Suppose* $\mathsf{b} > \mathsf{j} + \mathfrak{r}$. *Put* $\mathcal{A} = \sigma + \mathsf{z} - \mathsf{x} + \mathsf{a} + 2\mathsf{b} - \frac{1}{5}\mathfrak{v} + \mathsf{E}$ *and*

$$\begin{aligned}
\mathsf{P}_0 &= \mathsf{b} - \mathsf{j} - \tfrac{1}{5}\mathfrak{v} + 2\mathsf{z} + \mathsf{x} + 2\mathsf{E},\\
\mathsf{P}_1 &= \mathsf{m} - \tfrac{1}{2}\mathsf{n} - \mathsf{j} - \mathsf{i} - \tfrac{1}{5}\mathfrak{v} + \tfrac{1}{2}(\mathsf{Z} + \mathsf{E}),\\
\mathsf{P}_2 &= \tfrac{1}{2}\mathsf{a} - \mathsf{j} - \tfrac{1}{2}\mathsf{i} - \tfrac{1}{5}\mathfrak{v} + \tfrac{3}{2}\mathsf{z} + \mathsf{x} + \mathsf{E},\\
\mathsf{P}_3 &= \tfrac{1}{2}(\sigma + \mathsf{m}) - \tfrac{3}{2}\mathsf{j} - \tfrac{1}{2}(\mathfrak{r} + \mathsf{i}) - \tfrac{1}{5}\mathfrak{v} + \tfrac{1}{2}\mathsf{z} + \tfrac{1}{2}\mathsf{x} + \tfrac{3}{2}\mathsf{E}.
\end{aligned}$$

If $\mathsf{P}_0, \mathsf{P}_2, \mathsf{P}_3 > 0$, *and* $\mathsf{P}_1 > 0$ *or* $\mathsf{z} + \mathsf{i} + 2\mathsf{b} + \mathsf{a} < \mathsf{C}$, *then the sum of* (57) *over the dyads of* l *and* f *saves* $Q^{\mathsf{u}+\delta'}$ *for every* e.

Proof. Fix l, f, e and one sign; separate the weights on dyads $u' \asymp U'$, $d_0 \asymp A_0 \le Q^{\mathsf{a}-\mathsf{E}+o(1)}$ (with a sign), $b \asymp B_0 \le Q^{\mathsf{b}+o(1)}$ (with a sign), $y_0 \asymp Y_0 = Q^{\mathsf{y}+o(1)}$. The sum is then $\sum_{u',d_0} \alpha(u',d_0) \sum_{y_0,b} \beta(y_0)\gamma(b) e_{u'}(\pm r y_0 \overline{F_0 e d_0 b})$ with $|\alpha|, |\beta|, |\gamma| \le Q^{o(1)}$, where α contains $\Theta(u')$ and all the dependence on the factorisations hk, and the inner sum is over $(b, u') = 1$. By Cauchy's inequality in (u', d_0), majorising the dyads by smooth nonnegative $W(u'/U')V(d_0/A_0)$, dropping the conditions $(r, u') = 1$ and (when e is not expanded, as in Lemma 90(i)) $(d_0, lf) = 1$ in this nonnegative majorant, and keeping $(F_0 e d_0, u') = 1$,

$$|\Sigma|^2 \le Q^{o(1)} U' A_0 \sum_{y_0,b,y_0',b'} \beta\gamma\overline{\beta'\gamma'} \sum_{(R_0 d_0, u')=1} W\Big(\frac{u'}{U'}\Big) V\Big(\frac{d_0}{A_0}\Big) e_{u'}\big(k\,\overline{R_0 d_0}\big),$$

with $k = \pm r(y_0 b' - y_0' b)$ and $R_0 = F_0 e b b'$, since $y_0\overline{b} - y_0'\overline{b'} \equiv (y_0 b' - y_0' b)\overline{bb'}$.

Diagonal. If $k = 0$ then $y_0 b' = y_0' b$; for each value $v \le 4Y_0 B_0$ of this product there are $\tau(v)^2$ quadruples, so these terms contribute $Q^{o(1)} U' A_0 \cdot U' A_0 Y_0 B_0$.

Off-diagonal. Group the terms with $k \ne 0$ by the sign of k and by $(|k|, R_0)$, with coefficient $b_{k,R_0} = \sum \beta\gamma\overline{\beta'\gamma'}$ over the quadruples with these values. At fixed R_0 there are divisor-many pairs (b, b'); for one such pair with $v = (b, b')$, the solutions (y_0, y_0') of $y_0 b' - y_0' b = k/(\pm r)$ differ by multiples of $(b/v, b'/v)$, so there are $O(1 + Y_0 v/B_0)$ of them. Bounding one factor of $\sum |b_{k,R_0}|^2$ by the largest fibre and using $\sum_{b,b' \asymp B_0} (b, b') \ll B_0^2 \log B_0$,

$$\|b_{k,R_0}\|_2 \ll Q^{o(1)} Y_0 B_0 \Big(1 + \frac{Y_0}{B_0}\Big)^{1/2} \le Q^{\mathsf{y}+\mathsf{b}+o(1)},$$

as $\mathsf{y} \le \mathsf{j} < \mathsf{b}$. Apply (Q1) with $S = 1$, $s = 1$, smooth variables $c = u'$, $d = d_0$, numerators $n = |k| \le K_0 = Q^{\mathfrak{r}+\mathsf{y}+\mathsf{b}+o(1)}$ and $r = R_0 \asymp Q^{\mathsf{z}+\mathfrak{i}+\mathsf{E}+2\mathsf{b}+o(1)}$; the phase is $e(n\overline{rd}/c)$ or its conjugate, and the coprimality is exactly $(R_0 d_0, u') = 1$. As $\mathfrak{r} + \mathsf{y} \le \mathsf{j} + \mathfrak{r} < \mathsf{b}$, the numerator scale is below the denominator scale, $R + K_0 \asymp R$, and (50) has the three terms of degrees

$$\mathsf{C} + \mathsf{R}' + \max(\mathsf{C}, \mathsf{R}' + \mathsf{d}), \qquad 2\mathsf{C} + \mathsf{d} + \mathsf{R}', \qquad 2\mathsf{d} + \mathsf{K}_0 + \mathsf{R}',$$

with $\mathsf{R}' = \mathsf{z} + \mathfrak{i} + \mathsf{E} + 2\mathsf{b}$ and $\mathsf{K}_0 = \mathfrak{r} + \mathsf{y} + \mathsf{b}$. The off-diagonal part of the expanded sum is therefore $\ll Q^{(\mathsf{y}+\mathsf{b})+\frac{1}{2}\max(\cdots)+o(1)}$.

Budget. We need $|\Sigma| \ll Q^{\mathsf{C}+\mathsf{d}+\mathsf{y}+\mathsf{b}-\mathsf{u}-\delta'}$, that is, the expanded sum must be $\ll Q^{\mathsf{C}+\mathsf{d}+2\mathsf{y}+2\mathsf{b}-2\mathsf{u}-2\delta'}$. Using $\mathsf{u} \le \mathsf{j} + \frac{1}{10}\mathfrak{v} - \mathsf{Z} - \mathsf{E}$ (we do not use the term $-\mathfrak{i}$), this exponent is at least $\mathcal{A}$. Subtracting the four bounds from $\mathcal{A}$ gives the margins: P_0 for the diagonal; P_1 for the first term when $\mathsf{R}' + \mathsf{d} \ge \mathsf{C}$, and

$\mathsf{P}_2 + \frac{1}{2}\mathsf{d} > \mathsf{P}_2$ when $\mathsf{R}' + \mathsf{d} < \mathsf{C}$; P_2 and P_3 for the other two. Summing over f and l in their dyads, and over the $O(Q^{o(1)})$ dyads and signs, proves the claim. □

Lemma 87 (Direct estimate with the marks kept)**.** *Fix the labels g, r, ℓ, the dyads of l, f and of the other lengths, and $V_0 = grlf$, $V_1 = (h, V_0)$, $V_2 = V_0/V_1$. The map*

$$(h, k) \longmapsto (V_1, S_0, C_0), \qquad h = V_1 S_0, \quad k = V_2 C_0, \quad u' = S_0 C_0,$$

is a bijection from the factorisations $hk = V_0 u'$ onto the triples with $V_1 \mid V_0$, $S_0 C_0 = u'$ and $(S_0, V_2) = 1$. Let $\mathsf{w} = [S_0]$, so that $\mathsf{w} \le \min(\mathsf{C}, \mathfrak{b})$. Then the sum of (57) over all l, f, e, V_1 in the given dyads, at fixed dyad of S_0, has raw exponent at most

$$\mathsf{Z} + \tfrac{1}{2}\big(\mathsf{y} + \mathfrak{b} + \mathsf{w} - \mathsf{z} - (\mathsf{E} - \mathsf{z})_+ + \max \mathcal{K}\big),$$

where $\mathcal{K}$ consists of the seven affine forms

$$\begin{aligned} &\mathsf{c} + \mathsf{w} + \mathsf{p} + \mathsf{q} \quad (\mathsf{p} \in \{\mathsf{R} + \mathsf{w}, \mathsf{N}\},\ \mathsf{q} \in \{\mathsf{c}, \mathsf{R} + \mathsf{d}\}), \\ &2\mathsf{c} + \mathsf{d} + \mathsf{w} + \tfrac{1}{2}(\mathsf{p} + \mathsf{R}) \quad (\mathsf{p} \in \{\mathsf{R} + \mathsf{w}, \mathsf{N}\}), \qquad 2\mathsf{d} + \mathsf{N} + \mathsf{R}. \end{aligned} \tag{58}$$

Since there are $Q^{\mathsf{Z}+o(1)}$ pairs (l, f), each with the budget Q^{B}, the estimate saves if $\frac{1}{2}(\mathsf{y} + \mathfrak{b} + \mathsf{w} - \mathsf{z} - (\mathsf{E} - \mathsf{z})_+ + \max \mathcal{K}) < \mathsf{B}$. It saves without the gain $(\mathsf{E} - \mathsf{z})_+$ under the same condition with this term omitted.

Proof. The bijection. Given (h, k), $V_1 = (h, V_0)$, $S_0 = h/V_1$ and $V_2 = V_0/V_1$ are coprime, and $S_0 k = V_2 u'$, so $S_0 \mid u'$; put $C_0 = u'/S_0$, then $k = V_2 C_0$. Conversely $(V_1 S_0, V_0) = V_1 (S_0, V_2) = V_1$. The condition $(S_0, V_2) = 1$ involves S_0 only, and $S_0 \le h$, $S_0 \mid u'$ give $\mathsf{w} \le \min(\mathsf{C}, \mathfrak{b})$.

The sum. Insert r into numerator and denominator: by (56) the phase is $e_{S_0 C_0}(\pm N_0 \overline{R_0 d_0})$ with

$$N_0 = r^2 y_0, \qquad R_0 = r F_0 e b = r \ell\, l\, e\, b,$$

and every condition that involves C_0 or d_0 is the single coprimality $(R_0 d_0, S_0 C_0) = 1$: indeed $(r\ell l, u') = 1$ and $(ebd_0, u') = 1$ are (56), and no other condition on $d = lfS_0C_0$ remains after the expansions in f and e. The mark weight at $k = V_2 C_0$ and the weights of $d = lfS_0C_0$ depend on C_0 and on the representation; on common dyads they separate by Fourier inversion in the logarithms of $V_2, l, f, S_0, C_0, e, d_0$, so that the weights of C_0 and d_0 become fixed smooth functions $g(C_0, d_0)$ as in (Q1), and all the rest enters the coefficient b_{N_0, R_0, S_0}, which collects the terms of all (l, f, e, V_1, y_0, b) in

the dyads with the given (N_0, R_0, S_0): the mark weight at $h = V_1 S_0$, the cut $(S_0, V_2) = 1$, the coefficient of y_0 and the cuts $(f y_0, l) = 1$, $(b, lf) = 1$. The sign of R_0 is moved into N_0. In particular no coefficient depending on the modulus is attached to C_0, and the sum over C_0 keeps its sign.

The coefficient norm. At fixed (N_0, R_0, S_0) the integer y_0 is fixed and (l, e, b) is a factorisation of $R_0/(r\ell)$, with divisor-many choices; then $e \mid lf$ forces $e/(e,l) \mid f$, and $e/(e,l) \geq e/l$ has degree at least $(\mathsf{E}-\mathsf{z})_+$, so f has $Q^{\mathsf{x}-(\mathsf{E}-\mathsf{z})_+ + o(1)}$ choices in its dyad; and $V_1 \mid grlf$ has divisor-many values. The number of all tuples is $Q^{\mathsf{z}+\mathsf{x}+\mathsf{y}+\mathsf{b}+\mathsf{w}+o(1)}$. Bounding the sum of the squares by the largest fibre times the number of tuples,

$$\|b\|_2^2 \leq Q^{\mathsf{z}+2\mathsf{x}+\mathsf{y}+\mathsf{b}+\mathsf{w}-(\mathsf{E}-\mathsf{z})_+ + o(1)}$$

after extracting the common size of the weights.

The spectral factor. Apply (Q1) with $c = C_0$ (degree $\mathsf{c} = \mathsf{C} - \mathsf{w}$), $d = d_0$ (d), $n = N_0$ (N), $r = R_0$ (R), $s = S_0$ (w). The three terms of $\mathcal{K}^2$ in (50) have the degrees

$$\mathsf{c} + \mathsf{w} + \max(\mathsf{R}+\mathsf{w}, \mathsf{N}) + \max(\mathsf{c}, \mathsf{R}+\mathsf{d}),$$
$$2\mathsf{c} + \mathsf{d} + \mathsf{w} + \tfrac{1}{2}\big(\max(\mathsf{R}+\mathsf{w}, \mathsf{N}) + \mathsf{R}\big), \qquad 2\mathsf{d} + \mathsf{N} + \mathsf{R},$$

whose maximum is $\max \mathcal{K}$. So the sum is

$$\ll Q^{\frac{1}{2}(\mathsf{z}+2\mathsf{x}+\mathsf{y}+\mathsf{b}+\mathsf{w}-(\mathsf{E}-\mathsf{z})_+ + \max\mathcal{K}) + o(1)},$$

which is the raw exponent stated, as $\frac{1}{2}(\mathsf{z}+2\mathsf{x}) = \mathsf{Z} - \frac{1}{2}\mathsf{z}$. Without pooling f (fixing f and summing Q^{x} bounds) the coefficient norm loses the term $-(\mathsf{E}-\mathsf{z})_+$ and the bound is the same with this term omitted. □

Lemma 88 (Reciprocal completion). *If* $\mathsf{u} < \min(\frac{1}{2}\mathsf{C},\ \mathsf{b} - \frac{1}{2}\mathsf{C})$, *then* (57) *saves* $Q^{\mathsf{u}+\delta'}$ *at each modulus* u', *for every* l, f, e.

Proof. Fix l, f, e, u', y_0, d_0 and a dyad $|b| \asymp B_0 \leq Q^{\mathsf{b}+o(1)}$, and expand $(b, lf) = 1$ as $\sum_{\nu \mid (b, lf)} \mu(\nu)$; terms with $(\nu, u') > 1$ vanish. For $b = \nu b_1$ the weight of b_1 is a smooth function $W(\nu b_1 / B_0)$ with bounded rescaled derivatives (after separation), and the summand is $e_{u'}(w_0 \overline{b_1}) \mathbf{1}_{(b_1, u')=1}$ with $w_0 \equiv \pm r y_0 \overline{F_0 e d_0 \nu}$. Poisson summation modulo u' gives

$$\sum_{b_1} W\Big(\frac{\nu b_1}{B_0}\Big) e_{u'}(w_0 \overline{b_1}) = \frac{B_0}{\nu u'} \sum_{h' \in \mathbb{Z}} \widehat{W}\Big(\frac{h' B_0}{\nu u'}\Big) S(h', w_0; u').$$

Since r, F_0, e, d_0, ν are units modulo u', $(h', w_0, u') \leq (y_0, u')$, and (Q4) gives $|S(h', w_0; u')| \leq Q^{o(1)} \sqrt{u'} (y_0, u')^{1/2}$ for every h', including $h' = 0$. So the

b-sum is $\ll Q^{o(1)}(y_0,u')^{1/2}(\sqrt{u'}+B_0/\sqrt{u'})$. As $y_0\ge 1$, $\sum_{y_0\le Y_0}(y_0,u')^{1/2}\le \sum_{v\mid u'} v^{1/2}Y_0/v\le \tau(u')Y_0$. Compared with the trivial bound $Q^{\mathsf{y}+\mathsf{b}}$ of the sum over (y_0,b), the saving is $\min(\frac12\mathsf{C},\mathsf{b}-\frac12\mathsf{C})$ for every u',d_0; the divisor sums over ν cost $Q^{o(1)}$. Since this is an estimate at each modulus, the dyads of l,f are summed trivially. □

Lemma 89 (Kuznetsov at the represented level). *Fix l,f,e and put $\mathsf{J}=\mathsf{C}-\mathsf{d}=\mathsf{n}-\mathsf{Z}+\mathsf{E}$ and $\mathsf{A}_*=\mathsf{R}+\mathsf{w}$.*

(i) *If $\mathsf{J}<0$, the sum (57) vanishes for all sufficiently large T.*

(ii) *If $\mathsf{J}\ge 0$, the sum of (57) over V_1 and over one dyad of S_0, in the parametrisation of Lemma 87, has raw exponent at most*

$$\begin{aligned}\mathsf{F}_*=\mathsf{d}+\tfrac12\mathsf{R}+\mathsf{b}+\mathsf{w}+\tfrac12(\mathsf{J}+\mathsf{y})\\ +\tfrac12\big((\mathsf{J}-\mathsf{A}_*)_+ +(\mathsf{N}-\mathsf{A}_*)_+\big)+\tfrac{7}{32}\big(\mathsf{C}+\mathsf{J}+\mathsf{N}-\tfrac32\mathsf{R}-2\mathsf{w}\big)_+.\end{aligned}\tag{59}$$

So the estimate saves if $\mathsf{J}<0$, or if $\mathsf{F}_*<\mathsf{B}$.

Proof. Fix the labels g,r,ℓ, the sign, l,f,e, the dyads, V_1 (hence V_2), S_0 and b. As in the proof of Lemma 87, the phase is $e_{S_0C_0}(\pm N_0\overline{R_0d_0})$ with $N_0=r^2y_0$ and $R_0=r\ell\,l\,e\,b$, and every condition on C_0 or d_0 is $(R_0d_0,S_0C_0)=1$. Here we do not separate the weight of d_0. With $d=lfS_0C_0$ it is $\widehat{W}_2(ed_0U_n/d)=\widehat{W}_2(d_0/D_0)$, $D_0=lfS_0C_0/(eU_n)$, and d_0 runs over all integers coprime to $q'=S_0C_0$ (the expansion $a=ed_0$ leaves no other condition on d_0). Put $T_0=q'/D_0=eU_n/(lf)$; it does not depend on C_0, and $[T_0]=\mathsf{E}+\mathsf{n}-\mathsf{z}-\mathsf{x}=\mathsf{J}$.

Poisson summation in d_0. Splitting d_0 into classes x modulo q' and applying Poisson summation to $k\mapsto\widehat{W}_2((x+kq')/D_0)$, whose Fourier transform is $W_2(-\cdot)$ by Fourier inversion,

$$\sum_{(d_0,q')=1}\widehat{W}_2\Big(\frac{d_0}{D_0}\Big)e_{q'}\big(\pm N_0\overline{R_0d_0}\big)=\frac{1}{T_0}\sum_{t\in\mathbb{Z}}W_2\Big(-\frac{t}{T_0}\Big)S\big(t,\pm N_0\overline{R_0};q'\big).$$

W_2 is supported in a fixed compact subset of $(0,\infty)$, so only $t\ne0$ with $-t\asymp T_0$ occur. If $\mathsf{J}<0$ then $T_0\to0$, no such t exists for large T, and the sum vanishes. This proves (i). Assume now $\mathsf{J}\ge0$.

The Kloosterman sums as sums over a cusp pair. Put $A=|R_0|S_0$. Since $(R_0,S_0)=1$, Lemma 42 applies to $\Gamma_0(A)$ with the exact divisor $|R_0|$ in place of a and S_0 in place of m; the moduli of the cusp pair $(\infty,\mathfrak{a})$, $\mathfrak{a}=1/S_0$, are

$\gamma = c\sqrt{|R_0|}$ with $S_0 \mid c$ and $(c, R_0) = 1$, that is $c = S_0C_0$ with $(C_0, R_0) = 1$, which is exactly our range of C_0. By (54), with $n' = \pm\operatorname{sgn}(R_0)\, r^2 y_0$,

$$S\big(t, \pm N_0\overline{R_0}; S_0C_0\big) = S\big(t, n'\,\overline{|R_0|}; S_0C_0\big) = S_{\infty\mathfrak{a}}\big(t, n'; S_0C_0\sqrt{|R_0|}\big).$$

All remaining weights of C_0 (the mark weight at $k = V_2C_0$, the separated weights of $d = lfS_0C_0$, including the factor d^{-3w} from $V_{\mathfrak{a}}$, and $\widehat{W_1}(bU_m/d)$) form a smooth function $\Psi(C_0)$ supported on $C_0 \asymp Q^{\mathsf{c}+o(1)}$, with $(C_0\partial_{C_0})^i\Psi \ll_i Q^{o(1)}\|\Psi\|_\infty$; the weights of y_0 form a coefficient $\beta(y_0) \ll Q^{o(1)}$. Write $\Psi(C_0) = \gamma^{-1}\phi(4\pi\sqrt{|tn'|}/\gamma)$ with $\gamma = S_0C_0\sqrt{|R_0|}$, and remove the dependence of ϕ on $|tn'|$ by Mellin inversion in $|t|/T_0$ and $|n'|/(r^2Y_0)$, where $Y_0 = Q^{\mathsf{y}+o(1)}$ is the scale of y_0, on fixed enlarged intervals, as after Lemma 42; the resulting functions ϕ are supported on $\xi \asymp \xi_0$ with

$$[\xi_0] = \tfrac12(\mathsf{J}+\mathsf{N}) - \big(\mathsf{C} + \tfrac12[R_0]\big), \qquad \Phi_0 := \sup|\phi| = Q^{\mathsf{C}+\frac12[R_0]+o(1)}\|\Psi\|_\infty,$$

and $\xi^i\phi^{(i)} \ll_i Q^{o(1)}\Phi_0$, and the Mellin factors $(|t|/T_0)^{iv_1}$, $(|n'|/r^2Y_0)^{iv_2}$ go into the coefficients $\alpha_t = W_2(-t/T_0)(|t|/T_0)^{iv_1}$ and $\beta'_{n'} = \beta(y_0)(|n'|/r^2Y_0)^{iv_2}$. Put $\mathsf{X} = \mathsf{C} + \frac12\mathsf{R} - \frac12(\mathsf{J}+\mathsf{N})$; at the full b-scale $[R_0] = \mathsf{R}$ and $-[\xi_0] = \mathsf{X}$. By the identity $2\mathsf{X} = 2 - \varsigma + \mathfrak{v}$ (§6), and since in general $[R_0] \ge \mathsf{R} - \mathsf{b} \ge \mathsf{R} - 1$, we have $-[\xi_0] \ge \mathsf{X} - \frac12 \ge \frac13$, so $\xi_0 \le 1$ for large T.

Spectral expansion. By (53) with $m' = t$,

$$\sum_{C_0}\Psi(C_0)S_{\infty\mathfrak{a}}(t, n'; \gamma) = \sum_j \overline{\rho_{j\infty}(t)}\rho_{j\mathfrak{a}}(n')\widetilde{\phi}(t_j) + (\text{holomorphic terms}) + (\text{Eisenstein terms}),$$

and the transforms satisfy $\widetilde{\phi}(t) \ll_A Q^{o(1)}\Phi_0(1+|t|)^{-A}$ for real t, $\widetilde{\phi}(k) \ll_A Q^{o(1)}\Phi_0 k^{-A}$, and $\widetilde{\phi}(i\tau) \ll Q^{o(1)}\Phi_0\xi_0^{-2\tau}$ for $0 < \tau \le \theta = \frac{7}{64}$. Sum over t and y_0 with the coefficients α_t, $\beta'_{n'}$. For the regular spectrum we use Cauchy's inequality on each dyadic range of $|t_j|$ (or of k) and (51) at level A for the cusps ∞ (indices $|t| \asymp T_0$) and $\mathfrak{a}$ (indices $|n'| \asymp Q^{\mathsf{N}}$, supported on multiples of r^2); with the rapid decay of the transforms this gives

$$\ll Q^{o(1)}\Phi_0\Big(1 + \frac{T_0}{A}\Big)^{1/2}\Big(1 + \frac{Q^{\mathsf{N}}}{A}\Big)^{1/2}\|\alpha\|_2\|\beta'\|_2.$$

For the exceptional spectrum put $Y_1 = \max(1, A^2/T_0^3)$ and $Y_2 = \max(1, A^2Q^{-3\mathsf{N}})$, and write $\xi_0^{-2\tau_j} = (\xi_0^{-2}/Y_1Y_2)^{\tau_j}\, Y_1^{\tau_j}Y_2^{\tau_j}$. The first factor is at most $\max(1, \xi_0^{-2}/Y_1Y_2)^\theta$. By Cauchy's inequality, and by (52) with $Y = Y_i$ when $Y_i > 1$ (its right side is then $\ll Q^{o(1)}$ times the norm, as noted after

(52)) or (51) with $U = 1$ when $Y_i = 1$, the exceptional terms are bounded by the same expression times $\max(1, \xi_0^{-2}/Y_1Y_2)^{\theta}$. Here

$$\Big[\frac{\xi_0^{-2}}{Y_1Y_2}\Big] \le 2\mathsf{X} - (2\mathsf{A}_* - 3\mathsf{J}) - (2\mathsf{A}_* - 3\mathsf{N}) = 2\big(\mathsf{C} + \mathsf{J} + \mathsf{N} - \tfrac{3}{2}\mathsf{R} - 2\mathsf{w}\big)$$

at the full b-scale, where $[A] = \mathsf{A}_*$ (the last equality is an identity checked in §6; for smaller $|b|$ see below), and $2\theta = \frac{7}{32}$.

Collecting. $\|\alpha\|_2^2 = Q^{\mathsf{J}+o(1)}$, $\|\beta'\|_2^2 \le Q^{\mathsf{y}+o(1)}$, $[A] = \mathsf{A}_*$ and $[\Phi_0] = \mathsf{C} + \frac{1}{2}\mathsf{R}$ at the full b-scale. Multiplying by $T_0^{-1} = Q^{-\mathsf{J}}$ and using $\mathsf{C} - \mathsf{J} = \mathsf{d}$, the sum at fixed (V_1, S_0, b) is $\ll Q^{\mathsf{F}_*-\mathsf{b}-\mathsf{w}+o(1)}$. There are divisor-many V_1, Q^{w} values of S_0 in its dyad and Q^{b} values of b, which gives (59). For $|b|$ in a smaller dyad, of degree $\mathsf{b}' < \mathsf{b}$, the same computation gives F_* with b replaced by b' in b and in R; its derivative in b' is at least $\frac{3}{2} - \frac{1}{2} - \frac{1}{2} - \frac{3}{2}\cdot\frac{7}{32} = \frac{11}{64} > 0$, so the full dyad is the worst case, and the $O(\log Q)$ dyads cost $Q^{o(1)}$. No bound depending on C_0 individually was used, and the sum over C_0 was kept with its signs and weights. □

Checks of the normalisation. We record the two points of Lemma 89 that depend on conventions. (1) Two cusps. In (53) both Maass coefficients carry $\cosh(\pi t_j)^{-1/2}$, as in (35); (51) and (52) are stated for exactly these coefficients, at the cusps ∞ and $\mathfrak{a} = 1/S_0$ of $\Gamma_0(|R_0|S_0)$, for which $\mu(\mathfrak{c}) = 1/A$; the geometric modulus is $S_0C_0\sqrt{|R_0|}$, so the scalar in front of ϕ is γ, not S_0C_0, and the Bessel argument is $4\pi\sqrt{|tn'|}/\gamma$; this is the source of $\frac{1}{2}\mathsf{R}$ in F_* and of X. The direction Kloosterman sums → spectrum has no diagonal term. (2) Conductor labels. By Lemma 82 the only dependence of the summand on d is the one listed after (55); after the bijection of Lemma 87 it is a smooth function of C_0, and the coprimality conditions are those of (56), which are exactly the conditions $(c, R_0) = 1$, $S_0 \mid c$ on the moduli of the cusp pair.

The case analysis

Lemma 90 (Covering). *Let $(\varsigma, \mathfrak{b}) \in \mathcal{P}^{+*}$, and consider a limiting tuple of the degrees of a datum, together with the degrees $\mathsf{z}, \mathsf{x}, \mathsf{E}, \mathsf{w}$ of l, f, e, S_0. Then one of the following has positive margin at the tuple: the trivial bound ($\tau < 0$, or $\mathsf{u} < 0$), Lemma 84, Lemma 86, Lemma 87, Lemma 88 and Lemma 89. More precisely:*

(i) *for $0 < \varsigma < \frac{1}{10}$ and every $\mathfrak{b}$, the trivial bound, Lemma 84 or Lemma 86 saves;*

(ii) *for $\frac{1}{10} \le \varsigma < \frac{1}{8}$ and $\mathfrak{b} < \frac{3-21\varsigma}{2}$, the trivial bound, Lemma 84, Lemma 86 or Lemma 87 (without the gain $(\mathsf{E} - \mathsf{z})_+$) saves;*

(iii) *on each of the bands* $\frac{1}{10} \le \varsigma < \frac{1}{7}$, $\frac{1}{7} \le \varsigma < \frac{1}{4}$, $\frac{1}{4} \le \varsigma < \frac{1}{3}$, *with the respective caps* $\mathfrak{b} < f(\varsigma) = \frac{2-7\varsigma}{4}$, $\frac{1-2\varsigma}{4}$, $\frac{1-3\varsigma}{2}$, *the trivial bound, Lemma 84, Lemma 88, Lemma 87 (with or without the gain) or Lemma 89 saves.*

Since $\frac{3-21\varsigma}{2} \ge \frac{2-7\varsigma}{4}$ exactly for $\varsigma \le \frac{4}{35} < \frac{1}{8}$, the three parts cover $\mathcal{P}^{+*}$.

Proof. Assume $\tau \ge 0$ and $\mathsf{u} \ge 0$ at the tuple (otherwise the trivial bound saves), and that Lemma 84 does not apply: $\mathsf{m} \le \frac{1}{2}\sigma+\tau$, and no split succeeds. We use $\lambda \le \mathsf{D} \le 1+\varsigma$ and $0 \le \mathsf{i} \le 3\mathfrak{g}+4\mathfrak{r}$ (Lemma 83), $3\mathfrak{r} \le 2\mathfrak{g}+3\mathfrak{r} \le \varsigma$ (from $\tau \ge 0$), and the identities below, all checked by expansion in §6:

$$\sigma - 4\tau = 1 - 4\varsigma + 7\mathfrak{g} + 11\mathfrak{r} + \tfrac{18}{5}\mathfrak{v}, \qquad \tfrac{2}{3}\lambda - \mathsf{B}_* = \tfrac{1}{6} - \tfrac{4}{3}\varsigma + \tfrac{5}{2}\mathfrak{g} + \tfrac{23}{6}\mathfrak{r} + \tfrac{17}{15}\mathfrak{v},$$
$$\lambda - \mathsf{B}_* - 4\tau = \tfrac{1}{2} - 5\varsigma + \tfrac{19}{2}\mathfrak{g} + \tfrac{29}{2}\mathfrak{r} + \tfrac{22}{5}\mathfrak{v}.$$

Let $d_1 \ge d_2 \ge d_3 \ge d_4$ be the four dual degrees, so $\mathsf{m} = d_1$, $\mathsf{n} = d_2$, $\mathsf{j} = d_3+d_4$, and put $\mathsf{U} = \mathsf{D} - \mathsf{B}_*$.

Step 1: gaps. Let $\varsigma < \frac{1}{8}$. Then $\sigma > 4\tau$, and every set G of the dual variables has degree $\deg G \le 2\tau$ or $\deg G \ge \mathsf{U}$. Indeed, if $2\tau < \deg G < \mathsf{U}$, then the complement has degree $\mathsf{D} - \deg G > \mathsf{B}_* \ge 2\tau$; if $\deg G$ is the smaller degree the split (G, G^c) has $\mathsf{s} = \deg G > 2\tau$ and $\mathsf{l} > \mathsf{B}_*$, and otherwise $\mathsf{s} = \mathsf{D} - \deg G > \mathsf{B}_* \ge 2\tau$ and $\mathsf{l} = \deg G > \mathsf{s} > \mathsf{B}_*$; either way the split succeeds. Moreover $3\mathsf{U} > \mathsf{D}$, since $\frac{2}{3}\mathsf{D} - \mathsf{B}_* \ge \frac{2}{3}\lambda - \mathsf{B}_* > 0$; so at most two variables have degree $\ge \mathsf{U}$.

Step 2: the square margins when $\mathsf{j} \le 2\tau$. Then $\mathsf{m} - \frac{1}{2}\mathsf{n} \ge \frac{1}{4}(\mathsf{m}+\mathsf{n}) = \frac{1}{4}(\mathsf{D}-\mathsf{j})$, $\mathsf{n} \le \frac{1}{2}(\mathsf{D}-\mathsf{j})$ and $\mathsf{b} = \sigma - \mathsf{m} \ge \frac{1}{2}\sigma - \tau$, and, with $\mathsf{D} - 4\mathsf{i} = \lambda - 3\mathsf{i} \ge \lambda - 3(3\mathfrak{g}+4\mathfrak{r}) \ge \lambda - 4(3\mathfrak{g}+4\mathfrak{r})$ and $\mathsf{D} \le 1+\varsigma$,

$$\mathsf{b} - \mathsf{j} - 2\mathfrak{r} - \tfrac{1}{5}\mathfrak{v} \ge \tfrac{1}{2}\sigma - 3\tau - 2\mathfrak{r} - \tfrac{1}{5}\mathfrak{v} = \tfrac{1}{2} - 3\varsigma + \tfrac{11}{2}\mathfrak{g} + \tfrac{13}{2}\mathfrak{r} + \tfrac{5}{2}\mathfrak{v},$$
$$4\mathsf{P}_1 \ge 1 - 9\varsigma + 5\mathfrak{g} + 10\mathfrak{r} + \tfrac{36}{5}\mathfrak{v} + 2(\mathsf{Z}+\mathsf{E}),$$
$$4\mathsf{P}_2 \ge 1 - 7\varsigma + 4\mathfrak{g} + 8\mathfrak{r} + \tfrac{23}{5}\mathfrak{v},$$
$$\mathsf{P}_3 \ge \tfrac{1}{2} - 3\varsigma + 4\mathfrak{g} + 6\mathfrak{r} + \tfrac{5}{2}\mathfrak{v}.$$

For $\varsigma < \frac{1}{8}$ the first line gives $\mathsf{b} > \mathsf{j} + \mathfrak{r}$ and $\mathsf{P}_0 \ge \mathsf{b} - \mathsf{j} - \frac{1}{5}\mathfrak{v} > 0$, and $\mathsf{P}_2, \mathsf{P}_3 > 0$; if moreover $\varsigma < \frac{1}{9}$, then $\mathsf{P}_1 > 0$ and Lemma 86 saves.

(i) Here $\varsigma < \frac{1}{10}$, so Step 1 applies and $\mathsf{U} - 4\tau \ge \lambda - \mathsf{B}_* - 4\tau > 0$. If $d_3 \ge \mathsf{U}$, then $\mathsf{D} \ge 3\mathsf{U}$, which is impossible; so $d_3, d_4 \le 2\tau$, and $\deg\{d_3, d_4\} = \mathsf{j} \le 4\tau < \mathsf{U}$ forces $\mathsf{j} \le 2\tau$ by Step 1. By Step 2, Lemma 86 saves.

(ii) Here $\frac{1}{10} \le \varsigma < \frac{1}{8}$ and $\mathsf{w} \le \mathfrak{b} < \frac{3-21\varsigma}{2}$. We use the margins $\mathsf{M}_\mathsf{k} = \mathsf{B} - \frac{1}{2}(\mathsf{y} + \mathsf{b} + \mathsf{w} - \mathsf{z} + \mathsf{k})$ of Lemma 87 without the gain, one for each $\mathsf{k} \in \mathcal{K}$. Expanding (§6),

$$\begin{aligned}
2\mathsf{M}_{(\mathsf{R}+\mathsf{w},\mathsf{c})} &= 1 - \varsigma + \mathfrak{g} + \mathfrak{r} + \tfrac{4}{5}\mathfrak{v} + (\mathsf{m} - \mathsf{n}) + (\mathsf{Z} - \mathsf{E}) + \mathsf{z} - \mathsf{w},\\
2\mathsf{M}_{(\mathsf{N},\mathsf{c})} &= 1 - 2\varsigma + 3\mathfrak{g} + 3\mathfrak{r} + \tfrac{9}{5}\mathfrak{v} + \mathsf{m} + 2\mathsf{x} + 3\mathsf{z},\\
2\mathsf{M}_{(\mathsf{N},\mathsf{R}+\mathsf{d})} &= 2\mathsf{m} + \mathsf{n} - 2\varsigma + 4\mathfrak{g} + 3\mathfrak{r} - \mathfrak{i} + \tfrac{9}{5}\mathfrak{v} + \mathsf{x} + \mathsf{z} - \mathsf{w},\\
2\mathsf{M}'_\mathsf{N} &= \mathsf{m} + \tfrac{1}{2}\mathsf{n} - \tfrac{3}{2}\varsigma + 3\mathfrak{g} + 3\mathfrak{r} + \tfrac{13}{10}\mathfrak{v} + \tfrac{3}{2}\mathsf{x} + \tfrac{5}{2}\mathsf{z} + \tfrac{1}{2}\mathsf{E},\\
\mathsf{M}_1 := \mathsf{M}_{(\mathsf{R}+\mathsf{w},\mathsf{R}+\mathsf{d})} &= \mathsf{m} - \tfrac{1}{2}\varsigma + \mathfrak{g} + \tfrac{1}{2}\mathfrak{r} + \tfrac{2}{5}\mathfrak{v} - \mathsf{w} - \tfrac{1}{2}\mathfrak{i} - \tfrac{1}{2}\mathsf{E},\\
2\mathsf{M}_2 := 2\mathsf{M}'_{\mathsf{R}+\mathsf{w}} &= \mathsf{m} - \varsigma + 2\mathfrak{g} + 2\mathfrak{r} + \tfrac{4}{5}\mathfrak{v} - \tfrac{1}{2}\mathsf{w} + \mathsf{Z} + \mathsf{z},\\
2\mathsf{M}_3 := 2\mathsf{M}'' &= 2(\mathsf{m} + \mathsf{n}) - 2\tau - 3\mathfrak{r} - \mathfrak{i} + \mathsf{E} - \mathsf{w},
\end{aligned}$$

where the subscript (p,q) refers to the first line of (58), M'_p to the second and M'' to the third. Below we always have $\mathsf{j} \le 4\tau$, so $\mathsf{m} + \mathsf{n} \ge \lambda - 4\tau \ge 1 - 3\varsigma$. With $\mathsf{m} \ge \mathsf{n}$, $\mathsf{E} \le \mathsf{Z}$, $\mathsf{w} \le \frac{1}{2}$, $4\mathfrak{g} + 3\mathfrak{r} - \mathfrak{i} \ge -\mathfrak{r} \ge -\frac{1}{3}\varsigma$, $2\mathsf{m} + \mathsf{n} \ge \frac{3}{2}(\mathsf{m} + \mathsf{n})$ and $\mathsf{m} + \frac{1}{2}\mathsf{n} \ge \frac{3}{4}(\mathsf{m} + \mathsf{n})$, the first four margins are at least $\frac{1}{2} - \varsigma$, $1 - 2\varsigma$, $1 - \frac{41}{6}\varsigma$ and $\frac{3}{4} - \frac{15}{4}\varsigma$, all positive. It remains to treat $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$.

Case 1: $\mathsf{j} \le 2\tau$. By Step 2, $\mathsf{P}_0, \mathsf{P}_2, \mathsf{P}_3 > 0$ and $\mathfrak{b} > \mathsf{j} + \mathfrak{r}$, and Lemma 86 saves unless $\mathsf{P}_1 \le 0$. Then $\varsigma \ge \frac{1}{9}$ and $\mathsf{E} \le \mathsf{Z} + \mathsf{E} \le \frac{1}{2}(9\varsigma - 1 - 5\mathfrak{g} - 10\mathfrak{r} - \frac{36}{5}\mathfrak{v})$. With $\mathsf{m} \ge \frac{1}{2}(\mathsf{m} + \mathsf{n}) \ge \frac{1}{2}(\lambda - 2\tau)$,

$$\begin{aligned}
\mathsf{M}_1 &\ge \tfrac{1}{2}(\lambda - 2\tau) - \tfrac{1}{2}\varsigma + \mathfrak{g} + \tfrac{1}{2}\mathfrak{r} + \tfrac{2}{5}\mathfrak{v} - \tfrac{1}{2}(3\mathfrak{g} + 4\mathfrak{r}) - \mathsf{w} - \tfrac{1}{2}\mathsf{E}\\
&\ge \tfrac{3}{4} - \tfrac{13}{4}\varsigma + \tfrac{5}{4}\mathfrak{g} + 2\mathfrak{r} + \tfrac{13}{5}\mathfrak{v} - \mathsf{w}\\
&> \tfrac{3}{4} - \tfrac{13}{4}\varsigma - \tfrac{3-21\varsigma}{2} = \tfrac{29\varsigma - 3}{4} > 0,\\
2\mathsf{M}_2 &\ge \tfrac{1}{2}(\lambda - 2\tau) - \varsigma - \tfrac{1}{2}\mathsf{w} \ge \tfrac{1}{2} - \tfrac{3}{2}\varsigma - \tfrac{1}{4} > 0,\\
2\mathsf{M}_3 &\ge 2(\lambda - 2\tau) - 2\tau - 3\mathfrak{g} - 7\mathfrak{r} - \mathsf{w} = 2 - 4\varsigma + 3\mathfrak{g} + 3\mathfrak{r} + \tfrac{17}{5}\mathfrak{v} - \mathsf{w} > 0.
\end{aligned}$$

So Lemma 87 saves.

Case 2: $\mathsf{j} > 2\tau$. By Step 1, $\mathsf{j} = \deg\{d_3, d_4\} \ge \mathsf{U}$. If $\mathsf{n} > 2\tau$, then also $\mathsf{n} \ge \mathsf{U}$ and $\mathsf{m} \ge \mathsf{U}$, and $\mathsf{D} \ge 3\mathsf{U}$, which is impossible; so $\mathsf{n} \le 2\tau$ and $\mathsf{j} \le 4\tau$. The sets $\{d_2, d_4\}$ and $\{d_2, d_3\}$ have degree at least $\mathsf{j} > 2\tau$, hence at least U, so $d_3, d_4 \ge \mathsf{U} - \mathsf{n}$ and $\mathsf{n} \ge d_3 \ge \mathsf{U} - 2\tau$. Hence

$$\mathsf{m} = \mathsf{D} - \mathsf{n} - \mathsf{j} \ge \lambda - 6\tau, \qquad \mathfrak{b} - \mathsf{j} = \sigma - \mathsf{D} + \mathsf{n} \ge \sigma - \mathsf{B}_* - 2\tau = \tfrac{1}{2}\sigma - 4\tau,$$

and $\mathfrak{b} - \mathsf{j} - 2\mathfrak{r} - \frac{1}{5}\mathfrak{v} \ge \frac{1}{2} - 4\varsigma + \frac{15}{2}\mathfrak{g} + \frac{19}{2}\mathfrak{r} + \frac{17}{5}\mathfrak{v} > 0$, which gives $\mathfrak{b} > \mathsf{j} + \mathfrak{r}$ and $\mathsf{P}_0 > 0$. With $\mathsf{a} = \sigma - \mathsf{n} \ge \sigma - 2\tau$,

$$\begin{aligned}
\mathsf{P}_1 &\ge 1 - 10\varsigma + 16\mathfrak{g} + 25\mathfrak{r} + \tfrac{87}{10}\mathfrak{v} + \tfrac{1}{2}(\mathsf{Z} + \mathsf{E}),\\
\mathsf{P}_2 &\ge \tfrac{1}{2} - 5\varsigma + 8\mathfrak{g} + \tfrac{25}{2}\mathfrak{r} + \tfrac{43}{10}\mathfrak{v} + \tfrac{3}{2}\mathsf{z} + \mathsf{x} + \mathsf{E},\\
\mathsf{P}_3 &\ge 1 - \tfrac{17}{2}\varsigma + \tfrac{29}{2}\mathfrak{g} + 22\mathfrak{r} + \tfrac{37}{5}\mathfrak{v} + \tfrac{1}{2}\mathsf{z} + \tfrac{1}{2}\mathsf{x} + \tfrac{3}{2}\mathsf{E}.
\end{aligned}$$

If these are positive, Lemma 86 saves. Otherwise, using $\mathsf{E} \le \mathsf{Z}$, one of $\mathsf{E} \le 10\varsigma - 1$, $\mathsf{E} \le 5\varsigma - \frac{1}{2}$, $\mathsf{E} \le \frac{1}{3}(17\varsigma - 2)$ holds. Now

$$\begin{aligned}\mathsf{M}_1 &\ge (\lambda - 6\tau) - \tfrac{1}{2}\varsigma + \mathfrak{g} + \tfrac{1}{2}\mathfrak{r} + \tfrac{2}{5}\mathfrak{v} - \tfrac{1}{2}(3\mathfrak{g} + 4\mathfrak{r}) - \mathsf{w} - \tfrac{1}{2}\mathsf{E} \\ &\ge 1 - \tfrac{11}{2}\varsigma - \mathsf{w} - \tfrac{1}{2}\mathsf{E} > 5\varsigma - \tfrac{1}{2} - \tfrac{1}{2}\mathsf{E},\end{aligned}$$

which is ≥ 0, $\ge \frac{1}{2}(5\varsigma - \frac{1}{2}) \ge 0$ and $\ge \frac{1}{6}(13\varsigma - 1) > 0$ in the three cases; and $2\mathsf{M}_2 \ge \lambda - 6\tau - \varsigma - \frac{1}{2}\mathsf{w} \ge \frac{3}{4} - 6\varsigma > 0$, $2\mathsf{M}_3 \ge 2\lambda - 10\tau - 3\mathfrak{r} - \mathsf{w} = 2 - 8\varsigma + 14\mathfrak{g} + 19\mathfrak{r} + 7\mathfrak{v} - \mathsf{w} > 0$. So Lemma 87 saves.

The affine reductions in (i) and (ii) are also checked by machine, independently of Steps 1 and 2: with the four dual degrees kept separately, all seven splits of the four variables allowed to fail and the powerful-part degree free, the failure of every estimate named in (i), respectively (ii), gives 13122, respectively 76545, linear systems, and a computer verifies exact certificates of the kind used in (iii) at 22, respectively 69, nodes of the corresponding disjunction tree and checks that these nodes cover every system. In (i) the program fixes $\mathsf{x} = 0$. This is not a monotone reduction for the alternative $\mathsf{z} + \mathsf{i} + 2\mathsf{b} + \mathsf{a} \ge \mathsf{C}$ of Lemma 86, because C decreases in x; that alternative is covered for every x by the lower bound for $4\mathsf{P}_1$ in Step 2.

(iii) If $\mathsf{J} < 0$, Lemma 89(i) saves; so assume $\mathsf{J} \ge 0$. Since $\mathsf{u} \ge 0$ gives $\mathsf{Z} \le \mathsf{j} + \frac{1}{10}\mathfrak{v}$ and $\mathsf{j} \le \frac{1}{2}\mathsf{D} \le \frac{1}{2}(1 + \varsigma)$, the identity $\sigma - \frac{1}{2}(1 + \varsigma) - \frac{1}{10}\mathfrak{v} = \frac{1}{2} - \frac{1}{2}\varsigma - \mathfrak{g} - \mathfrak{r} - \frac{1}{10}\mathfrak{v}$ (§6) and $\mathfrak{g} + \mathfrak{r} + \frac{1}{10}\mathfrak{v} \le \frac{1}{2}\varsigma$ give $\mathsf{C} \ge \frac{1}{2} - \varsigma > 0$; so all moduli u' are large. The hypotheses of an uncovered tuple are: all degrees nonnegative, $\tau \ge 0$, $\mathsf{m} \ge \mathsf{n}$, $2\mathsf{n} \ge \mathsf{j}$, $\mathsf{y} \ge 0$, $\mathsf{E} \le \min(\mathsf{Z}, \mathsf{a})$, $\mathsf{w} \le \min(\mathsf{C}, \frac{1}{2})$, $\mathsf{m} \le \frac{1}{2}\sigma + \tau$, $\mathsf{u} \ge 0$, $\mathsf{i} \le 3\mathfrak{g} + 4\mathfrak{r}$ and $\mathsf{J} \ge 0$. Failure of the split $\{d_1, d_2\} \mid \{d_3, d_4\}$ and of the split $\{d_1\} \mid \{d_2, d_3, d_4\}$, of Lemma 88, of Lemma 87 without and with the gain $(\mathsf{E} - \mathsf{z})_+$, and of Lemma 89(ii) means that at least one entry in each of the following rows is nonnegative:

$$\begin{array}{l} 2\tau - \mathsf{j}, \quad \mathsf{B}_* - \mathsf{m} - \mathsf{n}; \\ 2\tau - \mathsf{m}, \quad 2\tau - (\mathsf{D} - \mathsf{m}), \quad \mathsf{B}_* - (\mathsf{D} - \mathsf{m}); \\ \mathsf{u} - \frac{1}{2}\mathsf{C}, \quad \mathsf{u} + \frac{1}{2}\mathsf{C} - \mathsf{b}; \\ \text{for one } \mathsf{k} \in \mathcal{K}, \text{ both } L_{\mathsf{k}} = \mathsf{y} + \mathsf{b} + \mathsf{w} - \mathsf{z} + \mathsf{k} - 2\mathsf{B} \text{ and } L_{\mathsf{k}} - \mathsf{E} + \mathsf{z}; \\ \text{one of the eight affine expansions of } 2\mathsf{F}_* - 2\mathsf{B}. \end{array} \tag{60}$$

(For the fourth row: failure with the gain means $L_{\mathsf{k}} \ge (\mathsf{E} - \mathsf{z})_+$ for some k, which implies both entries; for the fifth, each positive part in F_* is replaced by 0 or by its argument.) On the third band the first two rows may be replaced by the single alternative $4\tau - \sigma \ge 0$, under which Lemma 84 does not apply. For a band $[\varsigma_0, \varsigma_1)$ with cap f, append $\varsigma - \varsigma_0 \ge 0$, $\varsigma_1 - \varsigma - g_* \ge 0$ and $f(\varsigma) - \mathsf{w} - g_* \ge 0$; a tuple inside the band would have $g_* > 0$, since

$\mathsf{w} \le \mathfrak{b} < f(\varsigma)$. This gives $2 \cdot 3 \cdot 2 \cdot 7 \cdot 8 = 672$ linear systems on each of the first two bands and $7 \cdot 2 \cdot 7 \cdot 8 = 784$ on the third. For each of them a computer verifies, in exact rational arithmetic, a nonnegative combination of the inequalities and of the constant 1 equal to $-g_*$ or to -1; so $g_* \le 0$, a contradiction. □

Remark (Without the represented level). Lemma 89 is the only place where (51), (52) and (53) are used. Without it, (i) and (ii) of Lemma 90 are unchanged, and in (iii) the hypothesis $\mathsf{J} \ge 0$, the second entry of the fourth row of (60) and the fifth row are dropped; the same computation verifies the $2 \cdot 3 \cdot 2 \cdot 7 = 84$ resulting systems on each of the bands $\frac{1}{10} \le \varsigma < \frac{1}{7}$ and $\frac{1}{7} \le \varsigma < \frac{1}{4}$ with the caps $\mathfrak{b} < \frac{13-47\varsigma}{30}$ and $\mathfrak{b} < \frac{7-19\varsigma}{30}$. With these caps in place of those of (iii), the argument below gives the coefficient

$$\frac{1767870158617164811674095218440187374694865783}{42458728008803622772455544218525820350000000} = 41.6373792038\ldots$$

(computed in §6), which then depends only on (Q1) and (Q4) among the quoted estimates.

Proof of Proposition 81. By Lemma 83 it suffices to prove (a) or (b) for every datum $\mathfrak{D}$, with a $\delta' > 0$ that does not depend on $\mathfrak{D}$ or T. The degrees of all data lie in a compact set $\mathcal{X}$: $(\varsigma, \mathfrak{b})$ lies in the compact image $E^* \subset \mathcal{P}^{+*}$ of E, and all other degrees lie in $[0, 2]$. Adjoin the degrees $\mathsf{z}, \mathsf{x}, \mathsf{E}, \mathsf{w}$ of l, f, e, S_0, which also lie in $[0, 2]$. Each estimate of Lemma 90 has as margin a minimum of finitely many affine functions of these degrees, and the margins of Lemmas 87 and 89 (a maximum over the cases of (58) or over the positive parts) are continuous; so if one has a positive margin at a point it has margin at least half that value on a neighbourhood. By Lemma 90 every point has such an estimate, and by compactness there are $\delta_0 > 0$ and a finite assignment of estimates to the sets of a finite open cover such that at every point the assigned estimate has margin at least δ_0. Since the degrees of a datum at height T differ from points of $\mathcal{X}$ by $O(\epsilon) + o(1)$, we may take ϵ small and T large so that all dyadic, divisor, truncation and Mellin losses, all $Q^{o(1)}$ factors, and the deviation of the actual degrees from the cover are below $\frac{1}{2}\delta_0$.

For a datum whose limiting degrees lie in a set assigned to Lemma 84, or with $\tau < 0$, we prove (b), respectively (a), directly; the choice does not depend on $\mathfrak{p}$, nor on the modulus within the datum. Otherwise we apply Lemma 85 at every modulus of the datum, reduce the modulus as in (57), and split the result into the $Q^{o(1)}$ sums over the dyads of l, f, S_0 and the

divisors $e \mid lf$; for each of these sums the assigned estimate (Lemmas 86 to 89, or the trivial bound if $\mathsf{u} < 0$) gives the saving $Q^{\mathsf{u}+\delta_0/2}$, which is (a) for the datum. In Lemmas 87 and 89 the choice is made on whole dyads of the labels and of S_0, so no coefficient depending on the modulus is attached to the smooth variable C_0, and the signed sum over C_0 is kept with the weight $\overline{F}$ at $q = hk$. All estimates hold uniformly for $\boldsymbol{\sigma}$ in compact subsets of $(i\mathbb{R})^4$ and Φ in bounded subsets of $\mathcal{W}$, because they use only the size and the rescaled derivatives of the weights. Summing over the $(\log T)^{O(1)}$ data gives $\Sigma_F \ll T^{1-\eta}$. □

8.3 The mixed pairing

Proposition 91 (Mixed pairing)**.** *Let $c \in C_c^\infty(\mathbb{R}^3)$ be real with $\operatorname{supp} c \subset \mathcal{F}^{+\sharp}$, and let $b \in C_c^\infty(\mathbb{R}^3)$. Uniformly for Φ in bounded subsets of C_c^∞ with a fixed compact support in $(1,2)$,*

$$\langle P_b, P_c\rangle_\Phi = a_3 T L^9 \Big(\int \Phi\Big) \int_\Omega h_b h_c + o(TL^9).$$

Proof. We go through the proof of Proposition 78 and record, for each step, what it uses.

Reduction to elementary sources. This step of the proof of Proposition 58 uses only Lemma 57, Lemma 33, Lemma 31 and the fact that h_c vanishes unless $r \in I_c$. These hold for every real $c \in C_c^\infty((0,1)^3)$ supported in $1 < |a| < 2$, in particular on $\mathcal{F}^{+\sharp}$. So it suffices to prove (47) for $\boldsymbol{u} \in (i\mathbb{R})^3$ with distinct coordinates, and as there $\langle P_{\boldsymbol{u},\phi}, P_c\rangle_\Phi = \langle Z, P_c\rangle_\Phi + O(T^{-A})$ with $Z(s) = \prod_k \zeta(s + u_k/L)$.

Fourier inversion in the third coordinate. Put

$$\mathcal{L}_3 c(x;\beta) = \int c(x_1, x_2, a_3) e^{\beta(1-a_3)}\, \mathrm{d}a_3, \qquad F_\beta = \overline{\mathcal{L}_3 c(\cdot;\beta)}.$$

The function F_β is supported in the compact projection E of $\operatorname{supp} c$ to the first two coordinates, which lies in $\mathcal{P}^+$ because $\operatorname{supp} c \subset \mathcal{F}^{+\sharp}$; and $\int |\widehat{F_\beta}| \ll_j (1+|\beta|)^{-j}$, $|\mathcal{L}_3 c| \ll_j (1+|\beta|)^{-j}$, by integration by parts in a_3, as there. The representation of $\overline{P_c(s)}$ as an integral over β of $e^{-\beta}\overline{A_{F_\beta}(s)}$ times a finite m-sum, Lemma 19 for the m-sum, the functional equation and Stirling's formula are used exactly as in the proof of Proposition 78; none of them depends on the support of c. This gives

$$\langle Z, P_c\rangle_\Phi = \frac{1}{2\pi i} \int_{(0)} \Big(I_{F_\beta}(\boldsymbol{\sigma}(\beta)) + \mathcal{E}(\beta)\Big)\, \mathrm{d}\beta + O(T^{-A}),$$

where $\boldsymbol{\sigma}(\beta) = (u_1, u_2, u_3, \beta)$, with $|\mathcal{E}(\beta)| \ll \frac{1+|\beta|}{L}\|Z\|_\Phi \|\zeta(s+\beta/L)A_{F_\beta}\|_\Phi$ for $|\beta| \le L$.

Norms. By Hölder's inequality, Harper's bound [6] (which rests on the Riemann hypothesis, a theorem of Alpöge [19]) and Lemma 31(i), applied to the Fourier superpositions of A_{F_β}, we have $\|Z\|_\Phi \ll (TL^9)^{1/2}$, $\|\zeta(s+\beta/L)A_{F_\beta}\|_\Phi \ll_j (TL^9)^{1/2}(1+|\beta|)^{-j}$ and $|I_{F_\beta}(\boldsymbol{\sigma}(\beta))| \ll_j TL^9(1+|\beta|)^{-j}$ for $|\beta| \le LT/4$; these bounds use only $\int |\widehat{F_\beta}|$ and not the support. Hence $\int |\mathcal{E}| = o(TL^9)$, as there.

The limit. For each fixed $\beta \in i\mathbb{R}$, Proposition 80, applied to F_β (supported in the compact set $E \subset \mathcal{P}^+$) and $\boldsymbol{\sigma}(\beta)$, gives $I_{F_\beta}(\boldsymbol{\sigma}(\beta)) = a_3 TL^9(\int \Phi)\int \mathcal{L}_3 c(x;\beta) S(\boldsymbol{\sigma}(\beta);x)\,\mathrm{d}x + o(TL^9)$; this is the only place where $\mathcal{P}^+$ enters. By dominated convergence, with the majorant $TL^9(1+|\beta|)^{-j}$,

$$\langle Z, P_c\rangle_\Phi = a_3 TL^9\Big(\int \Phi\Big)\frac{1}{2\pi i}\int_{(0)}\int_{\mathbb{R}^2} \mathcal{L}_3 c(x;\beta) S(\boldsymbol{\sigma}(\beta);x)\,\mathrm{d}x\,\mathrm{d}\beta + o(TL^9).$$

The contour shift to $\Re\beta = 2r_0$ uses only the polynomial bound for $S(\boldsymbol{\sigma}(\beta);x)$ on compact sets of x and the decay of $\mathcal{L}_3 c$, and identifies the double integral with Λ' in the proof of Lemma 66. Part (b) of that proof needs only that $c \in C_c^\infty((0,1)^3)$ is real with $a_1 + a_2 < 1$ on its support; on $\mathcal{F}^{+\sharp}$ we have $a_1 + a_2 < \frac{3}{5}$. Hence $\Lambda' = \int_\Omega h_{e^{-\boldsymbol{u}\cdot a}} h_c$, which is (47). □

8.4 Proof of Theorem 79

Fix $\Phi \in \mathcal{W}$, and let ψ, η be real. For $\psi \in C_c^\infty(\mathcal{F}^+)$, Lemma 53 gives $h_{\mathsf{V}\psi} = \psi$, and Lemma 31(ii) gives $\|F_\psi\|_\Phi \ll (TL^9)^{1/2}$. The coefficients of F_ψ live at $\log N/L \in I_\psi \subset (1, \frac{3}{2})$, and those of $P_{(\mathsf{V}\psi)^J}$ at $\log N/L \in 3 - I_\psi \subset (\frac{3}{2}, 2)$. We prove the limits required by Definition 7 for all pairs involving $\mathcal{T}_{\mathcal{F}^+}$; pairs within $\mathcal{T}_5$ are covered by Theorem 52. Since $\mathcal{F} \subset \mathcal{F}^+$, the family $\mathcal{T}_{\mathcal{F}^+}$ contains $\mathcal{T}_{\mathcal{F}}$, and $\mathcal{T}_7 \supseteq \mathcal{T}_6$.

(a) Numerators. As in part (a) of the proof of Theorem 52, $\mathrm{N}_T(F_\psi) = \langle P_{c_0}, F_\psi\rangle_\Phi + O(T^{-A})$ with $c_0(a) = \varphi_1(|a|)\omega_0(a)$, where $\varphi_1 \in C_c^\infty((1,2))$ is 1 near I_ψ. By Proposition 91 with $c = \mathsf{V}\psi$ and $b = c_0$, $\mathrm{N}_T(F_\psi)/\mathcal{N}_T \to \langle h_{c_0}, \psi\rangle = \langle \varphi_1(r)v, \psi\rangle = \langle v, \psi\rangle$. Further $\mathrm{N}_T(\mathcal{J}F_\psi) = \overline{\mathrm{N}_T(F_\psi)}$ has the limit $\langle v, \psi\rangle = \langle v, \psi \circ J\rangle$, as $v \circ J = v$.

(b) Gram entries within $\mathcal{T}_{\mathcal{F}^+}$, and the witness norm. For real $\psi, \eta \in C_c^\infty(\mathcal{F}^+)$, Proposition 91 with $c = \mathsf{V}\psi$ and $b = \mathsf{V}\eta$ gives $\mathrm{G}_T(F_\eta, F_\psi)/\mathcal{N}_T \to \langle h_{\mathsf{V}\eta}, h_{\mathsf{V}\psi}\rangle = \langle \eta, \psi\rangle$. The case $\eta = \psi$ is the norm of the witness, $\|F_\psi\|_\Phi^2/\mathcal{N}_T \to \|\psi\|^2$; it needs no separate argument, because Proposition 91 places no condition on the support of b beyond smoothness

and compactness. Next $\mathrm{G}_T(\mathcal{J}F_\psi, \mathcal{J}F_\eta) = \overline{\mathrm{G}_T(F_\psi, F_\eta)}$ has the limit $\langle \psi, \eta\rangle = \langle \psi\circ J, \eta\circ J\rangle$. For the cross terms, Lemma 48 and the norm bound give $\mathrm{G}_T(\mathcal{J}F_\eta, F_\psi) = \langle P_{(\mathsf{V}\eta)^J}, F_\psi\rangle_\Phi + o(TL^9)$, and this is $O(T^{-A})$ by Lemma 57, since $3 - I_\eta \subset (\frac{3}{2}, 2)$ and $I_\psi \subset (1, \frac{3}{2})$ are disjoint. This agrees with $\langle \eta\circ J, \psi\rangle = 0$, since $J\mathcal{F}^+ \cap \mathcal{F}^+ = \emptyset$. The entry $\mathrm{G}_T(F_\psi, \mathcal{J}F_\eta)$ is the complex conjugate.

(c) Cross-pairings with $\mathcal{T}_3^\infty$. As in part (c) of the proof of Theorem 52, it suffices to show $\mathrm{G}_T(K, F_\psi)/\mathcal{N}_T \to \langle h_{F_K}, \psi\rangle$ and $\mathrm{G}_T(\mathcal{J}K, F_\psi)/\mathcal{N}_T \to \langle h_{F_K}\circ J, \psi\rangle$ for $K \in \{U, V, D_\phi\}$. For U and V this is Lemma 51 with W_ψ, I_ψ, $\psi^\flat$ replaced by F_ψ, I_ψ, ψ; its proof uses $\|F_\psi\|_\Phi \ll (TL^9)^{1/2}$ (Lemma 31(ii)), Lemma 57, the pairing of F_ψ with a smooth compact source (Proposition 91), and in Step 4 that ψ vanishes unless $r \in I_\psi$. For $K = D_\phi$, with $\operatorname{supp}\phi \subset (0, \theta)$ and $\theta < 1$, the profiles $\phi(r)v$ and $\phi(3-r)v$ vanish on $1 < r < 2$; $\mathrm{G}_T(D_\phi, F_\psi) = O(T^{-A})$ by Lemma 57, and $\mathrm{G}_T(\mathcal{J}D_\phi, F_\psi) = O(T^{-A})$ by the integration by parts in part (c) of Section 5.7 with $\rho = \max I_\psi < \frac{3}{2}$, since $\theta + \rho < 3$.

(d) Cross-pairings with marked tests. Let $\eta \in C_c^\infty(\mathcal{Y})$ be real. Every $a \in \operatorname{supp}\mathsf{V}\psi$ has $|a| \in I_\psi \subset (1, 2)$, so Proposition 38 gives $\mathrm{G}_T(F_\psi, W_\eta)/\mathcal{N}_T \to \langle \psi, \eta^\flat\rangle$, and $\mathrm{G}_T(W_\eta, F_\psi)$ is the conjugate. By Lemma 48 and $\|W_\eta\|_\Phi \ll (TL^9)^{1/2}$, $\mathrm{G}_T(F_\psi, \mathcal{J}W_\eta) = \overline{\langle P_{(\mathsf{V}\psi)^J}, W_\eta\rangle_\Phi} + o(TL^9)$; every $a \in \operatorname{supp}(\mathsf{V}\psi)^J$ has $|a| \in 3 - I_\psi \subset (1, 2)$, so Proposition 38 and Lemma 26 give the limit $\langle \psi\circ J, \eta^\flat\rangle = \langle \psi, \eta^\flat\circ J\rangle$. The entries with $\mathcal{J}F_\psi$ are the complex conjugates.

(e) Cross-pairings with $\mathcal{T}_\mathcal{R}$. For real $\eta \in C_c^\infty(\mathcal{R})$, Proposition 58 with $c = \mathsf{V}\eta$ and $b = \mathsf{V}\psi$ gives $\mathrm{G}_T(F_\psi, B_\eta)/\mathcal{N}_T \to \langle \psi, \eta\rangle$ (the supports of ψ and η may overlap). By Lemma 48, $\mathrm{G}_T(\mathcal{J}F_\psi, B_\eta) = \langle P_{(\mathsf{V}\psi)^J}, B_\eta\rangle_\Phi + o(TL^9)$, and Proposition 58 with $b = (\mathsf{V}\psi)^J$ and Lemma 26 give the limit $\langle \psi\circ J, \eta\rangle$ (which is 0). Likewise $\mathrm{G}_T(\mathcal{J}B_\eta, F_\psi)/\mathcal{N}_T \to \langle \eta\circ J, \psi\rangle$ by Lemma 48 and Proposition 91 with $c = \mathsf{V}\psi$ and $b = (\mathsf{V}\eta)^J$. The remaining entries are complex conjugates of these, by $\langle \mathcal{J}H, \mathcal{J}K\rangle_\Phi = \overline{\langle H, K\rangle_\Phi}$ and $\langle H, \mathcal{J}K\rangle_\Phi = \overline{\langle \mathcal{J}H, K\rangle_\Phi}$, and all the limits are real.

All limits in (a)–(e) hold for every $\Phi \in \mathcal{W}$. Hence $\mathcal{T}_7$ is faithful.

The span. The real $\psi \in C_c^\infty(\mathcal{F}^+)$ are dense in $L^2(\mathcal{F}^+)$ and the $\psi\circ J$ in $L^2(J\mathcal{F}^+)$, so $\mathcal{H}(\mathcal{T}_7) \supseteq L^2(\mathcal{F}^+ \cup J\mathcal{F}^+)$; with $v\mathbf{1}_{S_3\cup\mathcal{Z}\cup\mathcal{R}\cup J\mathcal{R}} \in \mathcal{H}(\mathcal{T}_5)$ we get $v\mathbf{1}_{U_7} = v\mathbf{1}_{S_3\cup\mathcal{Z}\cup\mathcal{R}\cup J\mathcal{R}} + v\mathbf{1}_{U_7\setminus(S_3\cup\mathcal{Z}\cup\mathcal{R}\cup J\mathcal{R})} \in \mathcal{H}(\mathcal{T}_7)$.

The measure. Since $\mathcal{F}^+ \subset \{1 < r < \frac{3}{2}\}$, as in the proof of Theorem 67 we have $\mu(U_7) = C_0 + 2\mu((N_0 \cap \mathcal{F}^+) \setminus \mathcal{R})$. In the coordinates (k, x, z) the set $(N_0 \cap \mathcal{F}^+) \setminus \mathcal{R}$ lies in $\frac{1}{2} < k < \frac{7}{12}$: for $k \geq \frac{7}{12}$ the cap $x_+(k) = \frac{3-5k}{2}$ is at most the lower face $\frac{1}{2}(k - \frac{1}{2})$ that N_0 imposes on x (from $z > \max(k - x, 1 - k)$ and $z < \frac{1}{2} + x$). On N_0 the upper face for z is $\frac{1}{2} + x$ and the lower face is

$\max(k-x, 1-k)$; outside $\mathcal{R}$ it is also $z \geq z_0(k) = \frac{18-28k}{7}$, which matters only for $k < \frac{11}{21}$, since $z_0(k) - (1-k) = \frac{11-21k}{7}$. The gain over $\mu(U_6)$ consists of three parts, integrated with the reflected density $\frac{9!}{2}(k-2x)^2(z-x)^2(z-k+x)^2$: the part of $N_0 \setminus (\mathcal{R} \cup \mathcal{F})$ with $\frac{1}{2} < k < \frac{10}{19}$, where every mark is admitted; the two cells with $\frac{10}{19} < k < \frac{8}{15}$ below the cap $a_0(k) = \frac{21-39k}{2}$; and the seven cells of Table 1.

Table 1: The seven cells of $(N_0 \cap \mathcal{F}^+) \setminus \mathcal{R}$ with $\frac{35}{66} < k < \frac{7}{12}$ beyond the cap $a_0(k) = \frac{21-39k}{2}$, with $a_1 = \frac{7-12k}{4}$, $a_2 = \frac{2-3k}{4}$, $a_3 = \frac{3-5k}{2}$. The upper face for z is $\frac{1}{2} + x$; the masses are 2μ of the cell (both reflected halves).

Range of k	Lower x	Upper x	Lower z	2μ
$(\frac{35}{66}, \frac{8}{15})$	$a_0(k)$	$a_1(k)$	$1-k$	$0.000809878521\ldots$
$(\frac{8}{15}, \frac{7}{13})$	$\frac{1}{2}(k-\frac{1}{2})$	$2k-1$	$k-x$	$0.000206853802\ldots$
$(\frac{8}{15}, \frac{7}{13})$	$2k-1$	$a_1(k)$	$1-k$	$0.002612978444\ldots$
$(\frac{7}{13}, \frac{6}{11})$	$\frac{1}{2}(k-\frac{1}{2})$	$2k-1$	$k-x$	$0.000484475814\ldots$
$(\frac{7}{13}, \frac{6}{11})$	$2k-1$	$a_2(k)$	$1-k$	$0.000292151447\ldots$
$(\frac{6}{11}, \frac{4}{7})$	$\frac{1}{2}(k-\frac{1}{2})$	$a_2(k)$	$k-x$	$0.001041043286\ldots$
$(\frac{4}{7}, \frac{7}{12})$	$\frac{1}{2}(k-\frac{1}{2})$	$a_3(k)$	$k-x$	$0.000022888772\ldots$
Total				$0.005470270089\ldots$

The cap a_0 meets a_1 at $k = \frac{35}{66}$, the cap a_2 meets $2k-1$ at $k = \frac{6}{11}$, and a_3 meets $\frac{1}{2}(k-\frac{1}{2})$ at $k = \frac{7}{12}$. The first part of the gain has mass $0.017403320153\ldots$, of which the strip $\frac{11}{21} < k < \frac{10}{19}$ contributes $\frac{6048414814819537375}{1666766405108292973184} = 0.003628831728\ldots$; the second has mass $\frac{15921115733351043355967}{2646442481410023750000000} = 0.006016044499\ldots$; the third is the total of Table 1. Exact integration gives $2\mu((N_0 \cap \mathcal{F}^+) \setminus \mathcal{R}) = C - C_0$ with

$$C = \frac{1126668325607502584615015940142157374558020 61}{27056964762462501481857964726258382399631 36},$$

and the seven cells of Table 1 add exactly

$$\frac{2687551102852874525001825036730168 01}{4913013542079045074107795959600000000 0} = 0.00547027008949\ldots.$$

These values are computed by exact iterated polynomial integration and, independently, by exact triangulation of each cell. The same computation checks that each cell lies in N_0 and below the cap of the estimate that covers it, that the cells meet neither $\mathcal{R}$ nor each other in positive measure, and that they exhaust the region defined directly by (48).

The passage to the liminf. Proposition 8 turns faithfulness into a bound for the sharp moment: for each $\epsilon > 0$ it chooses a finite combination of tests, applies the smoothed inequality (3), which holds for every $\Phi \in \mathcal{W}$ and all large T, with a $\Phi \le \mathbf{1}_{(1,2)}$ of integral at least $1-\epsilon$, at the heights $2^{-k}T$, $1 \le k \le K$, and sums. It uses no bound for $M_6(T)$; Harper's bound enters the proof of Theorem 79 only through the norm bounds of Lemmas 31 and 33 in (a)–(e) and in Proposition 91. Its second part uses only $v\mathbf{1}_{U_7} \in \mathcal{H}(\mathcal{T}_7)$ and $U_7 = JU_7$. Hence $\liminf_{T\to\infty} M_6(T)/(c_3 T L^9) \ge \mu(U_7) = C$. □

Computation. The certificates of (iii) and the computations of this section are checked by one program, which uses only SymPy and exact rational arithmetic and runs in under a minute. It also checks the case analyses of Lemma 90(i)–(ii) and every affine identity used in Lemmas 83 to 87, 89 and 90, computes the coefficient of the remark after Lemma 90, and recomputes the ceiling of Proposition 3 with $\mathcal{F}^+$ in its list. The certificates were found by floating-point linear programming and are checked exactly.

9 Proof of Theorem 2

By Theorem 79 the family $\mathcal{T}_7$ is faithful and $v\mathbf{1}_{U_7} \in \mathcal{H}(\mathcal{T}_7)$. Faithfulness means that the limits of Definition 7 hold for every $\Phi \in \mathcal{W}$. The first part of Proposition 8 turns these smoothed limits into a bound for the sharp moment: it applies (3), with a $\Phi \le \mathbf{1}_{(1,2)}$ of integral at least $1-\epsilon$, at the heights $2^{-k}T$ for $1 \le k \le K$, sums, and lets $\epsilon \to 0$ and $K \to \infty$. No upper bound for $M_6(T)$ is used in this step; Harper's bound, and with it the Riemann hypothesis [19], enters only through the norm bounds of Lemmas 31 and 33 in the proofs of faithfulness. The second part of Proposition 8 uses only $v\mathbf{1}_{U_7} \in \mathcal{H}$ and $JU_7 = U_7$, so

$$\liminf_{T\to\infty} \frac{M_6(T)}{c_3 T (\log T)^9} \;\ge\; 9!\,\|P_{\mathcal{H}(\mathcal{T}_7)} v\|^2 \;\ge\; \mu(U_7) = C.$$

This proves Theorem 2. □

For the record, the regions are nested as follows, and so are the closed spans of the families. (In passing from $\mathcal{T}_3$ to $\mathcal{T}_4$ the family $\mathcal{T}_3$ is replaced by its smooth part $\mathcal{T}_3^{\infty}$, which has the same closed span by Theorem 23; the assembly uses only the regions and the spans.)

Family	Region	Added by	μ
$\mathcal{T}_3$	S_3	Theorem 16	41.3163069650
$\mathcal{T}_4 = \mathcal{T}_3^{\infty} \cup \mathcal{T}_{\mathcal{M}}$	$S_3 \cup \mathcal{Z}$	Theorem 23	41.5917968750
$\mathcal{T}_5 = \mathcal{T}_4 \cup \mathcal{T}_{\mathcal{R}}$	$\cup\, \mathcal{R} \cup J\mathcal{R}$	Theorem 52	41.5958462794
$\mathcal{T}_6 = \mathcal{T}_5 \cup \mathcal{T}_{\mathcal{F}}$	$\cup\, \mathcal{F} \cup J\mathcal{F}$	Theorem 67	41.6117132746
$\mathcal{T}_7 = \mathcal{T}_5 \cup \mathcal{T}_{\mathcal{F}^+}$	$\cup\, \mathcal{F}^+ \cup J\mathcal{F}^+$	Theorem 79	41.6406029094

The missing mass $42 - C = 0.35939709059359626984\ldots$ is that of $(N_0 \setminus (\mathcal{R} \cup \mathcal{F}^+)) \cup J(N_0 \setminus (\mathcal{R} \cup \mathcal{F}^+))$. Inside N_0 the region $\mathcal{R}$ is limited by the trace condition $21k + 7r < 18$, which comes from the exponents in Heath-Brown's bounds (Lemma 56), and $\mathcal{F}^+$ by the mark caps (48), which are the conditions under which the case analysis of Lemma 90 closes. Both regions meet N_0 only in $\{k < \frac{7}{12}\}$; on $N_0 \cap \{k > \frac{7}{12}\}$, of mass $\frac{75263}{524288} = 0.1435527801\ldots$, no pairing beyond $\mathcal{T}_4$ is available.

Remark (A variance form of the bound). Let $D(x) = \sum_{n \le x} d_3(n)$ and

$$V(T) = \frac{T}{2c_3 (\log T)^9} \int_T^{T^{3/2}} \left[D(e^{2/T}x) - 2D(e^{1/T}x) + D(x) \right]^2 \frac{\mathrm{d}x}{x^2}.$$

In separate work in preparation, the identity $m_*(T) = 2 + 2V(T) + o(1)$ is derived, for a fixed smooth unit-mass time average $m_*(T)$ of $|\zeta(\frac{1}{2} + iTy)|^6/(c_3(\log T)^9)$; it uses the Riemann hypothesis [19] and a central-polynomial estimate that is not proved in this paper. Granting it, Theorem 2 gives

$$\begin{aligned} \liminf_{T \to \infty} V(T) \;&\ge\; \frac{C - 2}{2} \\ &= \frac{1072554396082577581651300010689640609758757 89}{5411392952492500296371592945251676479926272} \\ &= 19.8203014547\ldots. \end{aligned}$$

The conjecture $M_6(T) \sim 42\, c_3 T (\log T)^9$ of Conrey and Ghosh corresponds to $V(T) \to 20$. Neither 42 nor 20 is proved here.

10 The limit of support enlargement: proof of Proposition 3

Let $\mathcal{U}$ be the union of the sets listed in Proposition 3, that is, of $\mathcal{D}_+ = \{x + y < z\}$, $\{r < 1\}$, $\{x < \frac{1}{4}\}$, $\mathcal{R}$ and $\mathcal{F}^+$, and of their images under J. If every test in a faithful family has its profile in $L^2(\mathcal{U})$, then $\mathcal{H} \subseteq L^2(\mathcal{U})$, and

$$9!\, \|P_{\mathcal{H}} v\|^2 \le 9!\, \|P_{L^2(\mathcal{U})} v\|^2 = 9!\, \|v \mathbf{1}_{\mathcal{U}}\|^2 = \mu(\mathcal{U}).$$

So it suffices to show that $\mu(\mathcal{U})$ is the stated number. The set U_7 of Theorem 79 is contained in $\mathcal{U}$: by definition $\mathcal{D}_{\mathrm{acc}} \subseteq \{x < \frac14\} \cup J\{x < \frac14\}$ (if $x \ge \frac14$ and $\varrho \ge \frac14$ then $3\min(x,\varrho) + 5\max(x,\varrho) \ge 2$; and $x + y < \frac12$ forces $x < \frac14$), $\mathcal{A} = \{r < 1\} \cup J\{r < 1\}$, and $\mathcal{Z} \subseteq \mathcal{D}_+ \cup J\mathcal{D}_+$ (Section 5). Thus Proposition 3 bounds every extension of the present argument that adds tests only with profiles in these supports.

The complement. Since $J\mathcal{D}_+ = \{y + z - x > 1\}$, $J\{r < 1\} = \{r > 2\}$ and $J\{x < \frac14\} = \{z > \frac34\}$, the complement in Ω of the first three sets and their reflections is, up to a null set,

$$B = \{(x,y,z) \in \Omega :\ 1 < r < 2,\ x > \tfrac14,\ z < \tfrac34,\ x + y > z,\ y + z - x < 1\},$$

which is J-invariant. On $\mathcal{F}$ we have $x < \frac{k}{2} - \frac32(2k-1) = \frac32 - \frac52 k < \frac14$, as $k > \frac12$; hence $\mathcal{F} \cap B = \emptyset$, and $J\mathcal{F} \cap B = J(\mathcal{F} \cap B) = \emptyset$.

The larger region $\mathcal{F}^+$ does meet B, in the part of $\{\frac12 < k < \frac{10}{19}\}$ with $x > \frac14$ (where every mark is admitted), and only there: for $k \ge \frac{10}{19}$ the cap (48) is below $\frac{9}{38} < \frac14$. Since $\mathcal{F}^+ \subset \{r < \frac32\}$, $J\mathcal{F}^+ \subset \{r > \frac32\}$ and $\mathcal{R} \subset \{r < \frac{15}{14}\}$, adding $\mathcal{F}^+$ to the list increases $\mu(\mathcal{U})$ by

$$\begin{aligned} 2\mu\big((\mathcal{F}^+ \cap B) \setminus \mathcal{R}\big) &= \frac{19681318103514741804709 9}{4746377770796662411918500000 00} \\ &= 4.1465974800\ldots \cdot 10^{-7}. \end{aligned}$$

This increment, and the value of $\mu(\mathcal{U})$ found below, have also been checked by computer, together with $\mu(B)$ and $2\mu(B \cap \mathcal{R})$.

As $\mathcal{R} \cap J\mathcal{R} = \emptyset$ and J preserves μ and B,

$$\mu(\mathcal{U}) = 42 - \mu(B) + 2\mu(B \cap \mathcal{R}) + 2\mu\big((B \cap \mathcal{F}^+) \setminus \mathcal{R}\big).$$

The mass of B. Let $A(a,b,c) = (\frac{a+b}{2}, \frac{a+c}{2}, \frac{b+c}{2})$, with inverse $(x,y,z) \mapsto (x + y - z,\ x - y + z,\ y + z - x)$. It preserves the trace, has Jacobian $\frac14$, and halves the three pairwise differences, so $v \circ A = \frac18 v$ (as $v = \frac12 \Delta$) and $\mu(A(S)) = 2^{-8}\mu(S)$ for measurable $S \subseteq \Omega$. The conditions $x < y < z$, $x + y > z$ and $y + z - x < 1$ on $(x,y,z) = A(a,b,c)$ say that $0 < a < b < c < 1$, and $x > \frac14$, $z < \frac34$ say that $a + b > \frac12$, $b + c < \frac32$. Hence $B = A(B')$ with

$$\begin{aligned} B' &= \{(a,b,c) \in \Omega :\ 1 < a + b + c < 2\} \setminus (E' \cup JE'), \\ E' &= \{(a,b,c) \in \Omega :\ a + b \le \tfrac12,\ a + b + c > 1\}, \end{aligned}$$

because $J\{a + b \le \frac12\} = \{b + c \ge \frac32\}$, a point of E' has trace below $\frac32$, and $E' \cap JE' = \emptyset$ (on E', $b < \frac12$, so $b + c < \frac32$). By Section 4, $\mu(\{r < 1\}) =$

$\mu(\{r>2\})=1$, so the central band has mass 40. On E' put $k=a+b$; then $0<k<\frac12$, $0<a<\frac{k}{2}$ and $1-k<c<1$ (the conditions $b<c$ and $c<1$ being implied), and $9!v^2=\frac{9!}{4}(k-2a)^2(c-a)^2(c-k+a)^2$, so

$$\mu(E')=\frac{9!}{4}\int_0^{1/2}\int_{1-k}^{1}\int_0^{k/2}(k-2a)^2(c-a)^2(c-k+a)^2\,\mathrm{d}a\,\mathrm{d}c\,\mathrm{d}k=\frac{105}{8}.$$

Thus $\mu(B')=40-\frac{105}{4}=\frac{55}{4}$ and $\mu(B)=\frac{55}{1024}$.

The mass of $B\cap\mathcal{R}$. On $\mathcal{R}$ we have $k<\frac{18-7r}{21}$, and $x+y>z$ is $k>\frac{r}{2}$; these are compatible only for $r<\frac{36}{35}$. For $1<r<\frac{36}{35}$ and $\frac{r}{2}<k<\frac{18-7r}{21}$ the remaining conditions of B and Ω reduce to $\frac14<x<\frac{k}{2}$: indeed $z=r-k<\frac{r}{2}<\frac34$; $y<z$ is $x>2k-r$, and $2k-r<\frac{36-35r}{21}<\frac14$; and $y+z-x<1$ is $x>\frac{r-1}{2}$, with $\frac{r-1}{2}<\frac14$. In the coordinates (r,k,x) the x-integral is given by the formula in the proof of Theorem 52 with $t=k-\frac12$, and

$$2\mu(B\cap\mathcal{R})=\frac{134173355669}{14708889751500000000}.$$

Hence $42-\mu(B)+2\mu(B\cap\mathcal{R})=42-\frac{55}{1024}+\frac{134173355669}{14708889751500000000}=\frac{493586674117277079727}{11767111801200000000}$, and with the increment from $\mathcal{F}^+$ found above, $\mu(\mathcal{U})=\frac{13860404849860091}{330432202525696}$. Both masses are computed by exact rational triangulation. This computation integrates E' and B directly and checks $\mu(E')=\frac{105}{8}$, $\mu(B)=\frac{55}{1024}$ and the value of $2\mu(B\cap\mathcal{R})$. A second computation recomputes $2\mu(B\cap\mathcal{R})$ by iterated polynomial integration, taking $\mu(B)=\frac{55}{1024}$ as input. Any open subset of $B\setminus(\mathcal{R}\cup J\mathcal{R}\cup\mathcal{F}^+\cup J\mathcal{F}^+)$, for instance a small ball about $(0.30,0.45,0.50)$, which has trace $1.25\in(\frac{15}{14},\frac32)$ and $k=0.75>\frac35$, carries positive mass that no profile in $L^2(\mathcal{U})$ can see. □

Remark. A similar computation shows that the marked construction alone, granting every dominant-region pairing in both orientations, has ceiling $1339/32=41.84375$.

11 Formal verification in Lean

The finite parts of the proof have been checked in the Lean 4 proof assistant [37] with the Mathlib library [38]: the masses $\mu(\Omega)=42$ and $\mu(U_7)=C$, Proposition 3, the passage from faithful pairings to Theorem 2, and the covering Lemma 90. The analytic theorems that produce the pairings are not formalised. They enter as a single explicit hypothesis. The development uses version 4.29.1 of Lean and the matching release of Mathlib, and is available at `https://github.com/charlieoneill11/maths-releases`.

Statements. The formal statements define $M_6(T)$ with Mathlib's Riemann zeta function, the factor a_3 as a product over primes, and C as the rational number of Theorem 2, and state Theorem 2 as the assertion that $M_6(T) \geq c\, c_3 T(\log T)^9$ for all large T, for every $c < C$. They define the chamber Ω, the function v, the measure μ, the region U_7 and the region of Proposition 3, following Sections 2, 4 to 8 and 10. They also define tests, admissible weights, the smoothed numerators and Gram pairings, faithful families (Definition 7) and closed spans. The hypothesis asserts that some faithful family has $v\mathbf{1}_{U_7}$ in its closed span. This is the conclusion of Theorem 79, which rests on Theorems 16, 23, 52 and 67. Lean proves the following.

- $\mu(\Omega) = 42$ and $\mu(U_7) = C$, exactly.
- The region of Proposition 3 has mass $13860404849860091/330432202525696 < 42$, and $9!\|P_{\mathcal{H}} v\|^2$ is at most this mass for every family whose profiles vanish outside it. This is Proposition 3.
- Theorem 2, from the hypothesis above. Its proof formalises Proposition 8 and Section 9.
- Lemma 90 and the covering of the mark region that follows from it, for the affine exponent constraints of Section 8.

The formal statements were checked against the paper before the proofs were written. One correction was made afterwards: they had required the weights to be real analytic, which with their support condition left no nonzero weight, and they now require them to be infinitely differentiable, as in Section 2.

Proof. Lean proves the integration formula for polynomials over cells bounded by affine faces, and computes $\mu(\Omega)$, $\mu(U_7)$ and the mass of the region of Proposition 3 by cutting each region into such cells, up to null affine boundaries. The seven cells of Table 1 are among them, and their masses agree exactly with the rational values computed by the programs. For Lemma 90, Lean checks the affine identities of Section 8 by kernel computation, and the branches of part (iii) by Farkas certificates with nonnegative rational multipliers. These are the certificates used in the proof of Lemma 90, and a checker proved correct in Lean verifies them.

Trust. Except for Lemma 90, all of the above depends only on the three standard axioms of Lean, namely propositional extensionality, the axiom of choice and the soundness of quotients, with every computation carried out in the kernel. The proof of Lemma 90 also uses three evaluations by Lean's `native_decide` tactic, which runs compiled code, so accepting it means trusting the Lean compiler for those evaluations, which take a few seconds,

as well as the Lean kernel. The development contains no unproved steps and declares no further axioms.

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